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pETNNs: Partial Evolutionary Tensor Neural Networks for Solving Time-Dependent Partial Differential Equations

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Abstract. We present partial evolutionary tensor neural networks (pETNNs), a novel approach for solving time-dependent partial differential equations with high accuracy and capable of handling high-dimensional problems. Our architecture incorporates tensor neural networks and evolutionary parametric approximation. A posteriori error bound is proposed to support the extrapolation capabilities. In numerical implementations, we adopt a partial update strategy to achieve a significant reduction in computational cost while maintaining precision and robustness. Notably, as a low-rank approximation method of complex dynamical systems, pETNNs enhance the accuracy of evolutionary deep neural networks and empower computational abilities to address high-dimensional problems. Numerical experiments demonstrate the superior performance of the pETNNs in solving complex time-dependent equations, including the incompressible Navier-Stokes equations, high-dimensional heat equations, high-dimensional transport equations, and dispersive equations of higher-order derivatives.

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1 Introduction

Partial differential equations (PDEs) are ubiquitous in modeling phenomena across scientific and engineering disciplines. They serve as indispensable tools in modeling con-

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tinuum mechanics, electromagnetic theory, quantum mechanics, and a myriad of other fields where the evolution of systems across space and time is of interest. Traditional numerical approaches for solving PDEs, such as finite difference [31], finite element [8], and spectral methods [41], have been widely used. However, the computational burden imposed by these methods grows exponentially with increasing dimensionality of the problem, often rendering them impractical for high-dimensional systems. This phenomenon, known as the "curse of dimensionality", has been a persistent impediment to progress in various scientific domains.

The emergence of machine learning has introduced a novel set of tools to the scientific community, offering a potential panacea to the curse of dimensionality. Deep learning, a class of machine learning characterized by deep neural networks (DNNs), has been particularly successful in areas where traditional algorithms falter due to the complexity and volume of the data involved, such as [10, 17, 25, 30, 35, 54]. The universal approximation theorem underpins this capability, suggesting that a neural network can approximate any continuous function to the desired degree of precision [9,22]. Leveraging this, researchers have proposed various frameworks in which DNNs are trained to satisfy the differential operators, initial conditions, and boundary conditions of PDEs.

A notable advancement in the field is the emergence of deep Galerkin method [43], deep Ritz method [12], and physics-informed neural networks (PINNs) [39]. They embed the governing physical laws, encapsulated by PDEs, into the architecture of deep learning models. By incorporating the PDEs directly into the loss function, PINNs ensure that the learned solutions are not only data-driven but also conform to the underlying physical principles. This integration of physical laws into the learning process imbues PINNs with the ability to generalize beyond the data they were trained on, making them particularly adept at handling scenarios where data is scarce or expensive to acquire.

However, the efficacy of PINNs is predominantly limited to the temporal domain for which they have been trained, typically within the interval $[0,T]$. Their ability to extrapolate beyond this training window is limited, which is a manifestation of neural networks' inherent weakness in out-of-distribution generalization. This limitation hinders their predictive capacity, making them less effective in forecasting future states of the system under study.

The evolutionary deep neural networks (EDNNs) [11], which can address this challenge, have been developed as an innovative approach to solving time-dependent PDEs. The EDNNs are designed to evolve in tandem with the temporal dynamics they model, thus possessing an enhanced capability for prediction. This is achieved by structuring the neural network in a way that intrinsically accounts for the temporal evolution, allowing for a more robust extrapolation into future times. The methodology derived by the EDNNs has attracted significant attention. Hao *et al.* [19] proposed a neural energy descent method, which identifies steady-state solutions of evolutionary equations to optimize neural networks. The work [15] formulated the deep neural network parameters as an optimal control problem to approximate solution operators of evolutionary PDEs. The authors in [5] employed this to build upon the foundational results established in [1].