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A New $\mathcal{L}1$-TFPM Scheme for the Singularly Perturbed Subdiffusion Equations

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Abstract. Since the memory effect is taken into account, the singularly perturbed subdiffusion equation can better describe the diffusion phenomenon with small diffusion coefficients. However, near the boundary configured with non-smooth boundary values, the solution of the singularly perturbed subdiffusion equation has a boundary layer of thickness $\mathcal{O}(\varepsilon)$, which brings great challenges to the construction of the efficient numerical schemes. By decomposing the Caputo fractional derivative, the singularly perturbed subdiffusion equation is formally transformed into a class of steady-state diffusive-reaction equation. By means of a kind of tailored finite point method (TFPM) scheme for solving steady-state diffusion-reaction equations and the $\mathcal{L}1$ formula for discretizing the Caputo fractional derivative, we construct a new $\mathcal{L}1$-TFPM scheme for solving singularly perturbed subdiffusion equations. Our proposed numerical scheme satisfies the discrete extremum principle and is unconditionally numerically stable. Besides, we prove that the new TFPM scheme can obtain reliable numerical solutions as $h \ll \varepsilon$ and $\varepsilon \ll h$. However, there will be a large error loss due to the resonance effect as $h \sim \varepsilon$. Numerical experimental results can demonstrate the validity of the numerical scheme.

AMS subject classifications: 65M12, 65M15

Key words: Singularly perturbed subdiffusion equations, semi-discrete TFPM scheme, $\mathcal{L}1$-TFPM scheme, discrete extremum principle.

1 Introduction

Since the 19th century, the fractional derivatives are gradually used for modifying traditional physical models. Nigmatullin [16] uses the fractional Fick law to replace the

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classical Fick's law and then obtain the subdiffusion equation. The subdiffusion equation obtained from the modeling takes the memory effect into account, thus can model the universal electromagnetic, acoustic and mechanical responses more accurately. In recent decades, many scholars have devoted themselves to the theoretical analysis and numerical solution of the subdiffusion equation [2,8,10,13,14,17,20,23,24].

In this paper, we study the numerical approximation for the one-dimensional singularly perturbed subdiffusion equation on a bounded domain. We consider the following initial-boundary value problem on $\Omega_B^T=[-1,1]\times(0,T]$:

$$\begin{cases} {}_0^C D_t^\alpha u(x,t)-\varepsilon^2\partial_x\big(a(x)\partial_x\big)u(x,t)=f(x,t), & (x,t)\in\Omega_B^T,\\ u(-1,t)=\phi(t),\quad u(1,t)=\psi(t), & 0\le t\le T,\\ u(x,0)=w(x), & x\in[-1,1], \end{cases} \tag{1.1}$$

where $0<\varepsilon\ll 1$ is a small parameter and the diffusion coefficient $a(x)\in C_\infty^1([-1,1])$ satisfies the uniform ellipticity condition, that is, there are two constants $0<a_1<a_2<+\infty$ such that

$$\bar{a}_1\le a(x)\le\bar{a}_2,\quad \forall x\in[-1,1]. \tag{1.2}$$

Here, the Caputo fractional derivative ${}_0^C D_t^\alpha w(t)$ of order $\alpha\in(0,1)$ in time direction is defined as follow:

$${}_0^C D_t^\alpha w(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}w'(s)ds,$$

where $\Gamma(z)$ is the Gamma function. Besides, we assume the initial value condition $w(x)\in C_\infty^2([-1,1])$ and the initial boundary value condition satisfies the following compatibility conditions:

$$\phi(0)=w(-1),\quad \psi(0)=w(1). \tag{1.3}$$

We also assume that the source term $f(x,t)\in C_\infty^2([-1,1],L_\infty((0,T]))$. The existence and uniqueness of the solution for the initial-boundary value problem (1.1) can refer to the work in [11].

According to the work in [12], near the boundary configured with non-smooth boundary values, the solution $u(x,t)$ of the initial-boundary value problem (1.1) has a boundary layer of thickness $\mathcal{O}(\varepsilon)$. Besides, the singularity is mainly concentrated in the boundary layers, and the solution $u(x,t)$ changes gently outside the boundary layers. The fine structure associated with the small parameters ε contained in the boundary layers brings great challenges to the construction of an effective numerical scheme. Scholars have paid attention to the numerical solution of the singularly perturbed subdiffusion equation with low diffusion coefficient [3,9,18,21,22].

The tailored finite point method adaptively selects the local interpolation function according to the characteristics of the problem to be solved. In this way, the fine structure of the solution related to the small parameter ε can be captured on a relatively coarse grid. The tailored finite point method has been successfully used for the numerical solution of many singularly perturbed problems, please refer to the literature [6]. In [22], a tailored finite point scheme is introduced to solve the singularly perturbed time-fractional