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A New $\mathcal{L}1$-TFPM Scheme for the Singularly Perturbed Subdiffusion Equations

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Abstract. Since the memory effect is taken into account, the singularly perturbed subdiffusion equation can better describe the diffusion phenomenon with small diffusion coefficients. However, near the boundary configured with non-smooth boundary values, the solution of the singularly perturbed subdiffusion equation has a boundary layer of thickness $\mathcal{O}(\varepsilon)$, which brings great challenges to the construction of the efficient numerical schemes. By decomposing the Caputo fractional derivative, the singularly perturbed subdiffusion equation is formally transformed into a class of steady-state diffusive-reaction equation. By means of a kind of tailored finite point method (TFPM) scheme for solving steady-state diffusion-reaction equations and the $\mathcal{L}1$ formula for discretizing the Caputo fractional derivative, we construct a new $\mathcal{L}1$-TFPM scheme for solving singularly perturbed subdiffusion equations. Our proposed numerical scheme satisfies the discrete extremum principle and is unconditionally numerically stable. Besides, we prove that the new TFPM scheme can obtain reliable numerical solutions as $h \ll \varepsilon$ and $\varepsilon \ll h$. However, there will be a large error loss due to the resonance effect as $h \sim \varepsilon$. Numerical experimental results can demonstrate the validity of the numerical scheme.

AMS subject classifications: 65M12, 65M15

Key words: Singularly perturbed subdiffusion equations, semi-discrete TFPM scheme, $\mathcal{L}1$-TFPM scheme, discrete extremum principle.

1 Introduction

Since the 19th century, the fractional derivatives are gradually used for modifying traditional physical models. Nigmatullin [16] uses the fractional Fick law to replace the

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classical Fick's law and then obtain the subdiffusion equation. The subdiffusion equation obtained from the modeling takes the memory effect into account, thus can model the universal electromagnetic, acoustic and mechanical responses more accurately. In recent decades, many scholars have devoted themselves to the theoretical analysis and numerical solution of the subdiffusion equation [2,8,10,13,14,17,20,23,24].

In this paper, we study the numerical approximation for the one-dimensional singularly perturbed subdiffusion equation on a bounded domain. We consider the following initial-boundary value problem on $\Omega_B^T=[-1,1]\times(0,T]$:

$$\begin{cases} {}_0^C D_t^\alpha u(x,t)-\varepsilon^2\partial_x\big(a(x)\partial_x\big)u(x,t)=f(x,t), & (x,t)\in\Omega_B^T,\\ u(-1,t)=\phi(t), \quad u(1,t)=\psi(t), & 0\le t\le T,\\ u(x,0)=w(x), & x\in[-1,1], \end{cases} \tag{1.1}$$

where $0<\varepsilon\ll 1$ is a small parameter and the diffusion coefficient $a(x)\in C_\infty^1([-1,1])$ satisfies the uniform ellipticity condition, that is, there are two constants $0<a_1<a_2<+\infty$ such that

$$\bar{a}_1\le a(x)\le\bar{a}_2, \quad \forall x\in[-1,1]. \tag{1.2}$$

Here, the Caputo fractional derivative ${}_0^C D_t^\alpha w(t)$ of order $\alpha\in(0,1)$ in time direction is defined as follow:

$${}_0^C D_t^\alpha w(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}w'(s)ds,$$

where $\Gamma(z)$ is the Gamma function. Besides, we assume the initial value condition $w(x)\in C_\infty^2([-1,1])$ and the initial boundary value condition satisfies the following compatibility conditions:

$$\phi(0)=w(-1), \quad \psi(0)=w(1). \tag{1.3}$$

We also assume that the source term $f(x,t)\in C_\infty^2([-1,1],L_\infty((0,T]))$. The existence and uniqueness of the solution for the initial-boundary value problem (1.1) can refer to the work in [11].

According to the work in [12], near the boundary configured with non-smooth boundary values, the solution $u(x,t)$ of the initial-boundary value problem (1.1) has a boundary layer of thickness $\mathcal{O}(\varepsilon)$. Besides, the singularity is mainly concentrated in the boundary layers, and the solution $u(x,t)$ changes gently outside the boundary layers. The fine structure associated with the small parameters ε contained in the boundary layers brings great challenges to the construction of an effective numerical scheme. Scholars have paid attention to the numerical solution of the singularly perturbed subdiffusion equation with low diffusion coefficient [3,9,18,21,22].

The tailored finite point method adaptively selects the local interpolation function according to the characteristics of the problem to be solved. In this way, the fine structure of the solution related to the small parameter ε can be captured on a relatively coarse grid. The tailored finite point method has been successfully used for the numerical solution of many singularly perturbed problems, please refer to the literature [6]. In [22], a tailored finite point scheme is introduced to solve the singularly perturbed time-fractional convective diffusion equation with constant diffusion coefficient. In [12], the singularly perturbed subdiffusion equation is transformed into a singularly perturbed diffusive-reaction equation by discretization of the Caputo fractional derivatives. Then the singularly perturbed diffusive-reaction equation is discretized by a class of TFPM scheme. The $\mathcal{L}1$-TFPM scheme proposed in [12] can degenerate to the scheme in [22] when the diffusion coefficient is constant.

In fact, the $\mathcal{L}1$-TFPM scheme proposed in [12] is unconditionally stable, but not unconditionally consistent. For example, if we select $h=C\varepsilon\tau^{\alpha/2}$, as $h,\tau\to 0^+$, the difference equation discretized by this scheme will approximate the following partial differential equation:

$$ {}_0^C D_t^\alpha u(x,t)-\varepsilon^2\lambda(x)\partial_x\big(\lambda(x)a(x)\partial_x\big)u(x,t)=f(x,t) $$

with the coefficient $\lambda(x)$ defined by

$$ \lambda(x)=\frac{C}{\sqrt{a(x)\Gamma(2-\alpha)}\left[e^{\frac{C}{2\sqrt{\Gamma(2-\alpha)a(x)}}}-e^{-\frac{C}{2\sqrt{\Gamma(2-\alpha)a(x)}}}\right]}\neq 1. $$

Therefore, as $h\sim\varepsilon\tau^{\alpha/2}$, this scheme may not be able to obtain a reliable numerical solution for the singularly perturbed subdiffusion equation (1.1). We will construct a new class of $\mathcal{L}1$-TFPM scheme to overcome this deficiency.

By decomposing the Caputo fractional derivative, the singularly perturbed subdiffusion equation can be formally transformed into a family of steady-state diffusion-reaction equations with respect to the spatial variable x. By constructing a TFPM scheme for solving the steady-state diffusion-reaction equation, we can obtain a semi-discrete TFPM scheme for solving the singularly perturbed subdiffusion equation. The extremum principle is an important characteristic satisfied by the subdiffusion equation, which can ensure the stability of the solution. We can show that our proposed semi-discrete TFPM scheme satisfies the semi-discrete extremum principle. Furthermore, the semi-discrete scheme is unconditional stability and unconditionally consistent. It can be proved that when $\varepsilon\ll h$, the newly proposed semi-discrete TFPM scheme can still obtain high precision numerical solutions.

The solutions of ordinary differential equations discretized by the semi-discrete TFPM scheme cannot be given analytically. Since the solution $u(x,t)$ of the subdiffusion equation will have a weak singularity at the initial time [19], and we use the $\mathcal{L}1$ formula on the graded grids $\{t^k=T(k/M)^\gamma\}$ to approximate the Caputo fractional derivative in the ordinary differential equations obtained by discretization above. Thus, we can obtain a new $\mathcal{L}1$-TFPM scheme for solving the singularly perturbed subdiffusion equation. The newly proposed $\mathcal{L}1$-TFPM scheme is unconditionally stable and unconditionally consistent. The results of numerical experiments show that the newly proposed $\mathcal{L}1$-TFPM scheme can effectively solve the singularly perturbed subdiffusion equation as $\varepsilon\ll h$ and $h\ll\varepsilon$. However, as $h\sim\varepsilon$, the resonance effect results in a large loss of numerical accuracy.

This article is organized as follows. In Section 2, we propose a new $\mathcal{L}1$-TFPM scheme to numerically solve the singularly perturbed subdiffusion equation. In Section 3, we give some numerical experiments to show that the newly proposed $\mathcal{L}1$-TFPM scheme are valid. In Section 4, a brief summary is given.

2 The semi-discrete TFPM scheme and the $\mathcal{L}1$-TFPM scheme

In this section, we investigate the numerical solution of the singularly perturbed subdiffusion equation. We denote that the number of grids per unit length in spatial direction is N with a space step $h=1/N$ and $x_j=jh-1$, and the number of grids per unit length in time direction is M with a time step of $\tau=1/M$ and $t_n=n\tau$. Besides, we denote that $U_j(t)$ is a semi-discrete approximate value of $u(x_j,t)$ and Ω_h^T is the semi-dispersion of the domain Ω_B^T,

$$\Omega_h^T=\{(x_j,t)\,|\,-1\leq x_j\leq 1,0<t\leq T\}.$$

In addition, we denote that U_j^n is an approximate value of $u(x_j,t^n)$ and $\Omega_{h,\tau}^T$ is the grid division of the domain Ω_B^T,

$$\Omega_{h,\tau}^T=\left\{\left(x_j,t^n\right)\,|\,-1\leq x_j\leq 1,0\leq t^n\leq T\right\}.$$

2.1 The asymptotic analysis for the singularly perturbed subdiffusion equations

From the asymptotic analysis results proposed in [12], we know that the singularity of the solution $u(x,t)$ for the initial-boundary value problem (1.1) is concentrated in the space direction. Therefore, our research focuses on the numerical discretization of the spatial direction. Next, we briefly review the asymptotic analysis results of the singularly perturbed subdiffusion equation.

For convenience, we mark as follow:

$$u^{(o)}(x,t)=w(x)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(x,s)ds. \tag{2.1}$$

If the boundary value on the boundary is consistent with $u^{(o)}$, the boundary value is said to be smooth, otherwise it becomes a non-smooth boundary value. Through the asymptotic analysis results showed in [12], in general, we can find that the solution of (1.1) has rapidly changing boundary layers near the boundaries $\Gamma_\pm=\{\pm 1\}\times[0,T]$ configured with non-smooth boundary values.

Proposition 2.1 ([12]). *If a non-smooth boundary value is configured at the boundaries $\Gamma_\pm=\pm 1\times[0,T]$, the solution $u(x,t)$ of the initial boundary value problem* (1.1) *can be decomposed as follows:*

$$u(x,t)=u^{(o)}(x,t)+\bar{\theta}\left(\frac{x+1}{\varepsilon},t\right)+\bar{\eta}\left(\frac{1-x}{\varepsilon},t\right)+R(x,t). \tag{2.2}$$

Among them, the first item of decomposition is defined as in (2.1)*. However, the second and third terms of the decomposition satisfy the following estimates:*

$$\left|\bar{\theta}\left(\frac{1+x}{\varepsilon},t\right)\right| \leq C(\alpha)\gamma(t)\exp\left\{-\bar{E}(\alpha)\left[\frac{(1+x)^2}{\varepsilon^2 t^\alpha}\right]^{\frac{1}{2-\alpha}}\right\}, \tag{2.3}$$

$$\left|\bar{\eta}\left(\frac{1-x}{\varepsilon},t\right)\right| \leq C(\alpha)\gamma(t)\exp\left\{-\bar{E}(\alpha)\left[\frac{(1-x)^2}{\varepsilon^2 t^\alpha}\right]^{\frac{1}{2-\alpha}}\right\}, \tag{2.4}$$

where $C(\alpha)$ and $\bar{E}(\alpha)$ are positive constants independent of x and t, and the function $\gamma(t)$ is defined by

$$\gamma(t) = t^{\frac{\alpha}{4-3\alpha}} + \varepsilon t^{\frac{36-12\alpha-\alpha^2}{12-6\alpha}}.$$

Furthermore, the fourth term of the decomposition $R(x,t) = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0^+$.

Through asymptotic analysis, we can decompose the solution of the singularly perturbed subdiffusion equation (1.1) into three parts: the smooth part $u^{(0)}$, the singular part $\bar{\theta}+\bar{\eta}$ and the relatively small residual term R. The singular part $\bar{\theta}+\bar{\eta}$ shows that there are boundary layers of thickness $\mathcal{O}(\varepsilon)$ near the boundary $\Gamma_\pm$ configured with non-smooth boundary values. As $\varepsilon \ll 1$, the drastic changes of large gradients in the boundary layers will bring great challenges to the construction of efficient numerical methods. In addition, for the asymptotic property of $\partial_x^k u(x,t), k=1,2$, we need the following proposition:

Proposition 2.2 ([12])**.** *For the initial boundary value problem* (1.1)*, if the $k=1,2$ order derivative $\partial_x^k u(x,t)$ of its solution with respect to x exists and is continuous, we have the following estimates as $\varepsilon \to 0^+$:*

$$\left|\partial_x^k u(x,t)\right| \leq C(T,\alpha)\left[1+\varepsilon^{-2k} e^{-\bar{E}(\alpha)\left[\frac{(1+x)^2}{\varepsilon^2 t^\alpha}\right]^{\frac{1}{2-\alpha}}} + \varepsilon^{-2k} e^{-\bar{E}(\alpha)\left[\frac{(1-x)^2}{\varepsilon^2 t^\alpha}\right]^{\frac{1}{2-\alpha}}} + \varepsilon^2\right], \tag{2.5}$$

where the constants $C(T,\alpha), \bar{E}(\alpha)$ are independent of x,t and ε.

Remark 2.1. By the mathematical induction, we can generalize the result of the Proposition 2.2 to the spatial partial derivatives of any order $\partial_x^{(k)} u(x,t), k=1,2,3,\ldots$, that is, as $\varepsilon \to 0^+$,

$$\left|\partial_x^k u(x,t)\right| \leq C(T,\alpha)\left[1+\varepsilon^{-2k} e^{-\bar{E}(\alpha)\left[\frac{(1+x)^2}{\varepsilon^2 t^\alpha}\right]^{\frac{1}{2-\alpha}}} + \varepsilon^{-2k} e^{-\bar{E}(\alpha)\left[\frac{(1-x)^2}{\varepsilon^2 t^\alpha}\right]^{\frac{1}{2-\alpha}}} + \varepsilon^2\right], \tag{2.6}$$

where the constants $C(T,\alpha), \bar{E}(\alpha)$ are independent of x,t and ε.

2.2 The semi-discrete TFPM scheme

Using the idea of the line method, we firstly discretize the partial derivative of the spatial variable x in the singularly perturbed subdiffusion equation.

First, we decompose the Caputo fractional derivative ${}_0^C D_t^\alpha u(x,t)$ in the singularly perturbed subdiffusion equation into two parts

$$\begin{aligned}{}_0^C D_t^\alpha u(x,t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\partial_s u(x,s)ds \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t t^{-\alpha}\partial_s u(x,s)ds + \frac{1}{\Gamma(1-\alpha)}\int_0^t [(t-s)^{-\alpha}-t^{-\alpha}]\partial_s u(x,s)ds.\end{aligned}$$

For the first term of this decomposition, we have

$$\int_0^t t^{-\alpha}\partial_s u(x,s)ds = t^{-\alpha}[u(x,t)-u(x,0)] = t^{-\alpha}u(x,t) - t^{-\alpha}w(x), \quad t>0.$$

If we denote

$$\bar{f}(x,t) = f(x,t) + \frac{1}{\Gamma(1-\alpha)t^\alpha}w(x) - \frac{1}{\Gamma(1-\alpha)}\int_0^t [(t-s)^{-\alpha}-t^{-\alpha}]\partial_s u(x,s)ds,$$

the singularly perturbed subdiffusion equation can be rewritten as

$$-\varepsilon^2\partial_x\big(a(x)\partial_x\big)u(x,t) + \frac{1}{t^\alpha\Gamma(1-\alpha)}u(x,t) = \bar{f}(x,t), \quad (x,t)\in(-1,1)\times(0,T]. \tag{2.7}$$

Next, we will use the idea of the tailored finite point method showed in [6] to construct a semi-discrete numerical scheme for solving Eq. (2.7). Without loss of generality, we consider a tailored finite point scheme to solve the following differential equation on the grid $\{x_j | x_j = jh-1\}$:

$$-\bar{\varepsilon}^2\big(a(x)v'(x)\big)' + v(x) = g(x) \tag{2.8}$$

with the small parameter $\bar{\varepsilon}$ defined by

$$\bar{\varepsilon} = \bar{\varepsilon}(t,\alpha,\varepsilon) = \sqrt{\Gamma(1-\alpha)t^\alpha}\,\varepsilon.$$

At grid point x_j, we need to discretize the first term numerically in Eq. (2.8).

First of all, we define a new function $w(x)$ as follow:

$$w(x) = a(x)v'(x),$$

and then we need to examine the numerical approximations to $w'(x_j)$. Let us take the derivative of x on both sides of Eq. (2.8) and multiply both sides by $a(x)$. So, a simple calculation tells us that $w(x)$ satisfies the following differential equation:

$$-\bar{\varepsilon}^2 a(x)w''(x) + w(x) = \bar{g}(x) \tag{2.9}$$

with the force term defined by

$$\bar{g}(x)=a(x)g'(x).$$

On the interval $\mathbf{I}_j=[x_{j-1/2},x_{j+1/2}]$, we can use constants to approximate the diffusion coefficient $a(x)$ and the force term $\bar{g}(x)$ as follows:

$$a(x)\approx a(x_j)\triangleq a_j,\quad \bar{g}(x)\approx\bar{g}(x_j)\triangleq\bar{g}_j.$$

Hence, we can obtain an approximation of Eq. (2.8) on the small interval $\mathbf{I}_j$

$$-\bar{\varepsilon}^2 a_j w_h''(x)+w_h(x)=\bar{g}_j,\quad x\in\mathbf{I}_j=\left[x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}\right].$$

The general solution to the above differential equation can be written as

$$w_h(x)=Ae^{\frac{1}{\sqrt{a_j}\bar{\varepsilon}}x}+Be^{-\frac{1}{\sqrt{a_j}\bar{\varepsilon}}x}+(A+B)\bar{g}_j,\quad A,B\in\mathbb{R}. \tag{2.10}$$

At x_j, we can get

$$\begin{aligned} w_h'(x_j)&=\frac{1}{\sqrt{a_j}\bar{\varepsilon}}\left(Ae^{\frac{1}{\sqrt{a_j}\bar{\varepsilon}}x}-Be^{-\frac{1}{\sqrt{a_j}\bar{\varepsilon}}x}\right)\\ &=\frac{w_h\left(x_{j+\frac{1}{2}}\right)-w_h\left(x_{j-\frac{1}{2}}\right)}{\bar{h}_j}, \end{aligned} \tag{2.11}$$

where we denote a corrected step size $\bar{h}_j$ as follow:

$$\bar{h}_j=\bar{h}_j(t,\alpha,\varepsilon,h)=\sqrt{a_j}\bar{\varepsilon}\left(e^{\frac{h}{\sqrt{a_j}2\bar{\varepsilon}}}-e^{-\frac{h}{\sqrt{a_j}2\bar{\varepsilon}}}\right).$$

By our definition of $w(x)$, Eq. (2.9) can be rewritten as

$$-\bar{\varepsilon}^2 w'(x)+v(x)=g(x),$$

and then we can approximate $w'(x)$ at the grid point x_j by (2.11), that is,

$$\begin{aligned} w'(x_j)&\approx\frac{w\left(x_{j+\frac{1}{2}}\right)-w\left(x_{j-\frac{1}{2}}\right)}{\bar{h}_j}\\ &=\frac{a\left(x_{j+\frac{1}{2}}\right)v'\left(x_{j+\frac{1}{2}}\right)-a\left(x_{j-\frac{1}{2}}\right)v'\left(x_{j-\frac{1}{2}}\right)}{\bar{h}_j}\\ &=\frac{a_{j+\frac{1}{2}}v'\left(x_{j+\frac{1}{2}}\right)-a_{j-\frac{1}{2}}v'\left(x_{j-\frac{1}{2}}\right)}{\bar{h}_j}. \end{aligned} \tag{2.12}$$

Next, our research focuses on the numerical dispersion of $v'(x_{j\pm1/2})$. On the interval $\mathbf{I}_{j+1/2}=[x_j,x_{j+1}]$, we use constants to approximate $a(x)$ and $g(x)$ as follows:

$$a(x)\approx a\left(x_{j+\frac{1}{2}}\right)\triangleq a_{j+\frac{1}{2}},\quad g(x)\approx g\left(x_{j+\frac{1}{2}}\right)\triangleq g_{j+\frac{1}{2}}.$$

So, we approximate Eq. (2.8) on the interval $\mathbf{I}_{j+\frac{1}{2}}$ as follows:

$$-\bar{\varepsilon}^2 a_{j+\frac{1}{2}} v_h''(x)+v_h(x)=g_{j+\frac{1}{2}}, \quad x\in\mathbf{I}_{j+\frac{1}{2}}=\left[x_j,x_{j+1}\right].$$

Thus, the general solution of the above differential equation can be written as

$$v_h(x)=Ae^{\frac{1}{\sqrt{a_{j+1/2}}\bar{\varepsilon}}x}+Be^{-\frac{1}{\sqrt{a_{j+1/2}}\bar{\varepsilon}}x}+(A+B)g_{j+\frac{1}{2}}, \quad A,B\in\mathbb{R}. \tag{2.13}$$

Similar to (2.11), we can obtain,

$$v_h'\left(x_{j+\frac{1}{2}}\right)=\frac{v_h(x_{j+1})-v_h(x_j)}{\bar{h}_{j+\frac{1}{2}}}, \tag{2.14}$$

where we denote

$$\bar{h}_{j+\frac{1}{2}}=\bar{h}_{j+\frac{1}{2}}(t,\alpha,\varepsilon,h)=\sqrt{a_{j+\frac{1}{2}}}\,\bar{\varepsilon}\left(e^{\frac{h}{\sqrt{a_{j+1/2}}2\bar{\varepsilon}}}-e^{-\frac{h}{\sqrt{a_{j+1/2}}2\bar{\varepsilon}}}\right).$$

And then we can approximate $v'(x)$ at $x_{j+1/2}$ by (2.14) as follows:

$$v'\left(x_{j+\frac{1}{2}}\right)\approx\frac{v(x_{j+1})-v(x_j)}{\bar{h}_{j+\frac{1}{2}}}=\frac{v_{j+1}-v_j}{\bar{h}_{j+\frac{1}{2}}}. \tag{2.15}$$

Similarly, we can discretize $v'(x_{j-1/2})$ as follows:

$$v'\left(x_{j-\frac{1}{2}}\right)\approx\frac{v(x_j)-v(x_{j-1})}{\bar{h}_{j-\frac{1}{2}}}=\frac{v_j-v_{j-1}}{\bar{h}_{j-\frac{1}{2}}}. \tag{2.16}$$

In summary, at the grid point x_j, we can discretize the differential equation (2.9) numerically by the following tailored finite point method:

$$-\frac{\bar{\varepsilon}^2}{\bar{h}_j}\left(a_{j+\frac{1}{2}}\frac{v_{j+1}-v_j}{\bar{h}_{j+\frac{1}{2}}}-a_{j-\frac{1}{2}}\frac{v_j-v_{j-1}}{\bar{h}_{j-\frac{1}{2}}}\right)+v_j=g_j, \quad -1<x_j<1 \tag{2.17}$$

with the parameters $\bar{\varepsilon},\bar{h}_j$ and $\bar{h}_{j\pm1/2}$ defined as above. For convenience, we make the following notation:

$$\bar{\delta}_x v_{j+\frac{1}{2}}=\frac{v_{j+1}-v_j}{\bar{h}_{j+\frac{1}{2}}},$$

the tailored finite point scheme (2.17) can be rewritten as

$$-\bar{\varepsilon}^2\frac{a_{j+\frac{1}{2}}\bar{\delta}_x v_{j+\frac{1}{2}}-a_{j-\frac{1}{2}}\bar{\delta}_x v_{j-\frac{1}{2}}}{\bar{h}_j}+v_j=g_j, \quad -1\le x_j\le 1. \tag{2.18}$$

Applying the above numerical scheme to the discretization of the first term of (2.7), we can obtain a semi-discrete TFPM scheme to solve the initial-boundary value problem (1.1)

$$\begin{cases} {}_0^C D_t^\alpha U_j(t) - \frac{\varepsilon^2}{\bar{h}_j}\left[a_{j+\frac{1}{2}}\bar{\delta}_x U_{j+\frac{1}{2}}(t) - a_{j-\frac{1}{2}}\bar{\delta}_x U_{j-\frac{1}{2}}(t)\right] = f(x_j,t), & |x_j|<1, \quad 0<t\le T, \\ U_0(t)=\phi(t), \quad U_{2N+1}(t)=\psi(t), & 0<t\le T, \\ U_j(0)=w(x_j), & -1\le x_j\le 1, \end{cases} \tag{2.19}$$

where the notations $\bar{\delta}_x U_{j\pm 1/2}^n$ and $\bar{h}_j$ are defined as before. Let us define a vector function $\vec{\mathbf{U}}_h(t)$ as follows:

$$\vec{\mathbf{U}}_h(t) = [U_1(t), U_2(t), \cdots, U_{2N-2}, U_{2N-1}]^\top,$$

and then the semi-discrete TFPM scheme (2.19) can be rewritten as the following system of ordinary differential equations:

$$\begin{cases} {}_0^C D_t^\alpha \vec{\mathbf{U}}(t) + \mathbf{A}_h^\varepsilon(t)\vec{\mathbf{U}}(t) = \vec{\mathbf{F}}(t), \quad 0<t\le T, \\ \vec{\mathbf{U}}(0) = [w(x_1), w(x_2), \cdots, w(x_{2N-2}), w(x_{2N-1})]^\top, \end{cases} \tag{2.20}$$

where the stiffness matrix $\mathbf{A}_h^\varepsilon(t)$ is defined below

$$\mathbf{A}_h^\varepsilon(t) = \begin{bmatrix} \alpha_1^{\varepsilon,h}(t)+\beta_1^{\varepsilon,h}(t) & -\beta_1^{\varepsilon,h}(t) & & & \\ -\alpha_2^{\varepsilon,h}(t) & \alpha_2^{\varepsilon,h}(t)+\beta_2^{\varepsilon,h}(t) & -\beta_2^{\varepsilon,h}(t) & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha_{2N-2}^{\varepsilon,h}(t) & \alpha_{2N-2}^{\varepsilon,h}(t)+\beta_{2N-2}^{\varepsilon,h}(t) & -\beta_{2N-2}^{\varepsilon,h}(t) \\ & & & -\alpha_{2N-1}^{\varepsilon,h}(t) & \alpha_{2N-1}^{\varepsilon,h}(t)+\beta_{2N-1}^{\varepsilon,h}(t) \end{bmatrix}$$

with the coefficient equation $\alpha_j^{\varepsilon,h}(t), \beta_j^{\varepsilon,h}$ defined as below

$$\alpha_j^{\varepsilon,h}(t) = \frac{\varepsilon^2 a_{j-\frac{1}{2}}}{\bar{h}_j \bar{h}_{j-\frac{1}{2}}}, \quad \beta_j^{\varepsilon,h}(t) = \frac{\varepsilon^2 a_{j+\frac{1}{2}}}{\bar{h}_j \bar{h}_{j+\frac{1}{2}}}, \quad j=1,2,\ldots,2N-2,2N-1,$$

and the force term $\vec{\mathbf{F}}(t)$ is defined by

$$\vec{\mathbf{F}}(t) = \left[f(x_1,t)+\alpha_1^{\varepsilon,h}(t)\phi(t), f(x_2,t), \cdots, f(x_{2N-2},t), f(x_{2N-1},t)+\alpha_{2N-1}^{\varepsilon,h}(t)\psi(t)\right]^\top.$$

Remark 2.2. Different from the numerical scheme in [12], the corrected step size $\bar{h}_{j\pm 1/2}$ and $\bar{h}_j$ for the spatial direction used in our new scheme is independent of the time step τ, which avoids the inconsistency in some certain algebraic relations between h and τ.

Remark 2.3. As $h \ll \varepsilon$ and $t>0$, by the Taylor expansion, we can get

$$
\begin{aligned}
e^{\frac{h}{\sqrt{a_j}2\bar{\varepsilon}}} &= 1+\frac{h}{\sqrt{a_j}2\bar{\varepsilon}}+\frac{h^2}{8a_j\bar{\varepsilon}^2}+\mathcal{O}\left(\frac{h^3}{\bar{\varepsilon}^3}\right), \\
e^{-\frac{h}{\sqrt{a_j}2\bar{\varepsilon}}} &= 1-\frac{h}{\sqrt{a_j}2\bar{\varepsilon}}+\frac{h^2}{8a_j\bar{\varepsilon}^2}+\mathcal{O}\left(\frac{h^3}{\bar{\varepsilon}^3}\right).
\end{aligned}
$$

Thus, we have

$$
\bar{h}_j = h+\mathcal{O}\left(\frac{h^3}{\varepsilon^2}\right), \quad h \ll \varepsilon.
$$

Similarly, we can get

$$
\bar{h}_{j\pm\frac{1}{2}} = h+\mathcal{O}\left(\frac{h^3}{\varepsilon^2}\right), \quad h \ll \varepsilon.
$$

Hence, as $h \ll \varepsilon$, the given numerical scheme (2.19) will degenerate into the following traditional finite difference method (FDM) scheme:

$$
{}_0^CD_t^\alpha U_j(t)-\frac{\varepsilon^2}{h^2}\left[a_{j+\frac{1}{2}}\Delta_x U_{j+\frac{1}{2}}(t)-a_{j-\frac{1}{2}}\Delta_x U_{j-\frac{1}{2}}(t)\right]=f_j(t), \quad t>0, \tag{2.21}
$$

where we denote

$$
\Delta_x v_{j+\frac{1}{2}} = v_{j+1}-v_j, \quad \Delta_x v_{j-\frac{1}{2}} = v_j - v_{j-1}.
$$

Besides, as $h \to 0^+$, the difference equation (2.19) will approach the singularly perturbed subdiffusion equation (1.1).

2.3 The stability and convergence of the semi-discrete TFPM scheme

In [1, 15], a class of extremum principle for the subdiffusion equation over an open bounded domain is formulated and proved. Furthermore, the boundedness of the solution for the singularly perturbed subdiffusion equation can be obtained.

Proposition 2.3 ([1,15]). *Let $u(x,t)$ be a classical solution of the initial-boundary value problem* (1.1). *Assume the force term $f(x,t)\in C(\Omega)$, the boundary values $\phi(t),\psi(t)\in C([0,T])$ and the initial value $w(x)\in C([-1,1])$. If we denote*

$$
\begin{aligned}
&M_0 = \max\left\{\|w(x)\|_{C([-1,1])},\|\phi(t)\|_{C([0,T])},\|\psi(t)\|_{C([0,T])}\right\}, \\
&M_1 = \|f(x,t)\|_{C(\Omega)},
\end{aligned}
$$

then the following estimate of the solution norm holds true:

$$
\|u(x,t)\|_{C(\Omega)} \le M_0+\frac{T^\alpha}{\Gamma(1+\alpha)}M_1. \tag{2.22}
$$

Next, we show that the semi-discrete TFPM scheme (2.19) satisfies the semi-discrete extremum principle. Before that, we need the following lemma.

Lemma 2.1 ([1]). *Suppose that $f\in C^{(\alpha)}([0,T])$ and satisfies $f(t)\le f(t_0)$ $(t\le t_0)$ for some $t_0\in(0,T)$. Then we have*

$$
{}_0^C D_t^\alpha f(t_0)\ge 0. \tag{2.23}
$$

where

$$
C^{(\alpha)}([0,T]):=\left\{f\in C([0,T])\,\middle|\,{}_0^C D_t^\alpha f(t)\in C([0,T])\right\}.
$$

The semi-discrete maximum principle for the semi-discrete TFPM scheme (2.19) is given by the following proposition. The proof of this proposition is based on the idea for the continuous case proposed in [1,15].

Proposition 2.4 (Semi-Discrete Maximum Principle). *Let the grid function $\{U_j(t),|x_j|<1\}$ be a solution of the semi-discrete TFPM scheme* (2.19)*. If the force term $f_j(t)\le 0$, and then either $U_j(t)\le 0$ or the function $\{U_j(t)\}$ attains its positive maximum on the bottom or back-side parts $S_h^T=\{(x_j,0)\}_{j=0}^{2N}\bigcup\{x_0,x_{2N}\}\times(0,T]$, i.e.*

$$
U_j(t)\le\max\left\{0,\max_{(x_j,t)\in S_h^T}U_j(t)\right\},\quad \forall(x_j,t)\in\Omega_h^T=\{x_j\}_{j=0}^{2N}\times[0,T]. \tag{2.24}
$$

Proof. We prove this theorem by contradiction. We first suppose that the statement of the theorem does not hold true, that is, there exists a point $(x_{j_0},t_0),|x_{j_0}|<1,0<t_0\le T$ such that

$$
U_{j_0}(t_0)>\mathcal{M}=\max\left\{0,\max_{(x_j,t)\in S_h^T}U_j(t)\right\}>0.
$$

Let us introduce a number $\Delta=U_{j_0}(t_0)-\mathcal{M}$ and define an auxiliary grid function

$$
W_j(t)=U_j(t)+\frac{\Delta}{2}\frac{T-t}{T},\quad (x_j,t)\in\Omega_h^T.
$$

The following results can be easily obtained:

$$
W_j(t)\le U_j(t)+\frac{\Delta}{2},\qquad (x_j,t)\in\Omega_h^T, \tag{2.25}
$$

$$
\begin{aligned}
W_{j_0}(t_0)&\ge U_{j_0}(t_0)=\Delta+\mathcal{M}\ge\Delta+U_j(t)\\
&\ge\Delta+W_j(t)-\frac{\Delta}{2}\ge\frac{\Delta}{2}+W_j(t),\qquad (x_j,t)\in S_h^T.
\end{aligned} \tag{2.26}
$$

From the inequality (2.26), we can know that $W_j(t)$ cannot reach a maximum on S_h^T. If the maximum point of the grid function $\{W_j(t)\}$ over the grid Ω_h^T is denoted by $W_{j_1}(t_1)$ with $|x_{j_1}|<1,0<t_1\le T$ and

$$
W_{j_1}(t_1)\ge W_{j_0}(t_0)\ge\Delta+\mathcal{M}>\Delta.
$$

According to the result of Lemma 2.1, we can know that

$$ {}_0^C D_t^\alpha W_{j_1}(t_1) \geq 0. \tag{2.27} $$

Furthermore, we can easily get

$$ \begin{aligned} &-\left(a_{j_1+\frac{1}{2}}\bar{\delta}_x W_{j_1+\frac{1}{2}}(t_1) - a_{j_1-\frac{1}{2}}\bar{\delta}_x W_{j_1-\frac{1}{2}}(t_1)\right) \\ =& a_{j_1-\frac{1}{2}}\bar{\delta}_x W_{j_1-\frac{1}{2}}(t_1) - a_{j_1+\frac{1}{2}}\bar{\delta}_x W_{j_1+\frac{1}{2}}(t_1) \\ \geq& \frac{\bar{a}_1}{\bar{h}}\left(2W_{j_1} - W_{j_1+1} - W_{j_1-1}\right)(t_1) \geq 0. \end{aligned} $$

According to the definition of $W_j(t)$, we have

$$ U_j(t) = W_j(t) - \frac{\Delta}{2}\frac{T-t}{T}. $$

By simple calculation, we can get

$$ {}_0^C D_t^\alpha (t-T) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}. $$

Furthermore, we can get the following relation:

$$ {}_0^C D_t^\alpha U_j(t) = {}_0^C D_t^\alpha W_j(t) + \frac{\Delta}{2T}\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{2.28} $$

Additionally, we have

$$ \begin{aligned} &-\left(a_{j_1+\frac{1}{2}}\bar{\delta}_x U_{j_1+\frac{1}{2}} - a_{j_1-\frac{1}{2}}\bar{\delta}_x U_{j_1-\frac{1}{2}}\right)(t_1) \\ =& -\left(a_{j_1+\frac{1}{2}}\bar{\delta}_x W_{j_1+\frac{1}{2}} - a_{j_1-\frac{1}{2}}\bar{\delta}_x W_{j_1-\frac{1}{2}}\right)(t_1) \geq 0. \end{aligned} \tag{2.29} $$

Combining (2.27)-(2.29), we can obtain

$$ \begin{aligned} L_h^\varepsilon U_{j_1}(t_1) =& {}_0^C D_t^\alpha W_{j_1}(t_1) + \frac{\Delta}{2T}\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} \\ &- \frac{\varepsilon^2}{\bar{h}_{j_1}}\left(a_{j_1+\frac{1}{2}}\bar{\delta}_x W_{j_1+\frac{1}{2}} - a_{j_1-\frac{1}{2}}\bar{\delta}_x W_{j_1-\frac{1}{2}}\right)(t_1) > 0, \end{aligned} \tag{2.30} $$

where the discrete operator L_h^ε defined by

$$ L_h^\varepsilon U_j(t) = {}_0^C D_t^\alpha U_j(t) - \frac{\varepsilon^2}{\bar{h}_j}\left(a_{j+\frac{1}{2}}\bar{\delta}_x U_{j+\frac{1}{2}} - a_{j-\frac{1}{2}}\bar{\delta}_x U_{j-\frac{1}{2}}\right)(t). $$

On the other hand, by the conditions of the theorem, we have

$$ L_h^\varepsilon U_{j_1}(t_1) = f_{j_1}^{n_1} \leq 0, $$

which contradicts the inequality (2.30). Above all, the assumption made at the beginning of the theorem proof is wrong and thus this theorem is proved. ☐

Substituting $-U_j(t)$ instead of in the reasoning above, the semi-discrete minimum principle can be obtained.

Proposition 2.5 (Semi-Discrete Minimum Principle). *Let a grid function* $\{U_j(t),(x_j,t)\in\Omega_h^T\}$ *be a solution of the semi-discrete TFPM scheme* (2.19). *If the force term* $f_j(t)\geq 0$, *and then either* $U_j(t)\geq 0$ *or the function* $\{U_j(t)\}$ *attains its negative minimum on the bottom or back-side parts* $S_h^T=\{(x_j,0)\}_{j=0}^{2N}\bigcup\{x_0,x_{2N}\}\times(0,T]$, *i.e.*

$$U_j(t)\geq \min_{(x_j,t)\in S_h^T} U_j(t), \quad \forall (x_j,t)\in\Omega_h^T. \tag{2.31}$$

Based on the semi-discrete extremum principle above, we can prove that the semi-discrete TFPM scheme (2.19) is unconditionally stable.

Theorem 2.1 (Stability for the Semi-Discrete TFPM Scheme). *Let a grid function* $\{U_j(t)\}_{j=0}^{2N}$ *be a solution of the semi-discrete TFPM scheme* (2.19). *If we denote*

$$\begin{aligned} &M_0=\max\left\{\|w(x)\|_{C([-1,1])},\|\phi(t)\|_{C([0,T])},\|\psi(t)\|_{C([0,T])}\right\},\\ &M_1=\max_{0\leq j\leq 2N}\|f(x_j,t)\|_{C([0,T])}, \end{aligned}$$

and then the following estimation for the solution $\{U_j(t)\}_{j=0}^{2N}$ *holds true:*

$$\max_{j=0,\dots,2N}\|U_j(t)\|_{C([0,T])}\leq M_0+\frac{T^\alpha}{\Gamma(1+\alpha)}M_1, \quad 0<t\leq T. \tag{2.32}$$

Proof. To prove the theorem, we first introduce an auxiliary grid function $\{W_j(t)\}$,

$$W_j(t)=U_j(t)-\frac{M_1}{\Gamma(1+\alpha)}t^\alpha.$$

Then, we know

$$L_h^\varepsilon W_j(t)=L_h^\varepsilon U_j(t)-\frac{M_1}{\Gamma(1+\alpha)}{}_0^C D_t^\alpha t^\alpha\leq f_j(t)-M_1\leq 0.$$

Then, according to Proposition 2.4, we can get that for any $(x_j,t)\in\Omega_h^T$,

$$\begin{aligned} W_j(t)&\leq \max\left\{0,\max_{(x_j,t)\in S_h^T} W_j(t)\right\}\\ &\leq \max\left\{0,\max_{(x_j,t)\in S_h^T} U_j(t)-\frac{M_1}{\Gamma(1+\alpha)}t^\alpha\right\}\\ &\leq \max\left\{0,\max_{(x_j,t)\in S_h^T} U_j(t)\right\}\leq M_0. \end{aligned}$$

Furthermore, we can obtain for any $(x_j,t)\in\Omega_h^T$,

$$U_j(t)\le M_0+\frac{M_1}{\Gamma(1+\alpha)}t^{\alpha}\le M_0+\frac{M_1}{\Gamma(1+\alpha)}T^{\alpha}.$$

Besides, we first introduce another auxiliary grid function $\{V_j(t)\}$,

$$V_j(t)=U_j(t)+\frac{M_1}{\Gamma(1+\alpha)}t^{\alpha}.$$

It follows from a simple calculation to obtain

$$L_h^{\varepsilon}V_j(t)=L_h^{\varepsilon}U_j(t)+\frac{M_1}{\Gamma(1+\alpha)}{}_0^C D_t^{\alpha}t^{\alpha}\ge f_j(t)+M_1\ge 0.$$

Then, according to Proposition 2.5, we can get that for any $(x_j,t)\in\Omega_h^T$,

$$\begin{aligned}V_j(t)\ge\min_{(x_j,t)\in S_h^T}V_j(t)&\ge\min_{(x_j,t)\in S_h^T}\left\{U_j(t)+\frac{M_1}{\Gamma(1+\alpha)}t^{\alpha}\right\}\\&\ge\min_{(x_j,t)\in S_h^T}U_j(t)\ge -M_0.\end{aligned}$$

Furthermore, we can obtain for any $(x_j,t)\in\Omega_h^T$,

$$U_j(t)\ge -M_0-\frac{M_1}{\Gamma(1+\alpha)}t^{\alpha}\ge M_0-\frac{M_1}{\Gamma(1+\alpha)}T^{\alpha}.$$

In summary, the content of the theorem is proved. □

Remark 2.4. From the stability of the semi-discrete TFPM scheme (2.19), we can know that the numerical solution obtained by the numerical scheme (2.19) is unique and continuously depends on the initial-boundary values $\phi(t),\psi(t),w(x)$ and the source term $f(x,t)$.

Next, we examine the convergence of the semi-discrete scheme (2.19). The truncation error of the semi-discrete TFPM scheme at the internal points (x_j,t) is defined as below

$$\mathbb{T}_h u_j(t)={}_0^C D_t^{\alpha}u_j(t)-\frac{\varepsilon^2}{\bar{h}_j}\left(a_{j+\frac{1}{2}}\bar{\delta}_x u_{j+\frac{1}{2}}-a_{j-\frac{1}{2}}\bar{\delta}_x u_{j-\frac{1}{2}}\right)(t)-f_j(t),$$

where we denote

$$u_j(t)=u(x_j,t),\quad u_{j\pm\frac{1}{2}}(t)=u\big(x_{j\pm\frac{1}{2}},t\big).$$

In this paper, we hope that the truncation error $\mathbb{T}_h u_j(t)$ is small even when $0<\varepsilon\ll h$. It means that we used a coarse mesh comparing with the small parameter ε. Using the estimations in Propositions 2.1 and 2.2 for the solution $u(x,t)$ of the initial-boundary value problem (1.1), we have the following results:

Theorem 2.2 (Error Estimation for the Semi-Discrete TFPM Scheme). *As $0<\varepsilon \ll h$, that is, there exists a positive parameter γ_1 and a positive constant C_1 such that $0<\varepsilon\le C_1 h^{1+\gamma_1}$, if we define the error function $e_j(t)$ as follow:*

$$e_j(t)=u(x_j,t)-U_j(t),$$

and then we have the following error estimation:

$$\begin{aligned}&\max_{0\le j\le 2N}\|e_j(t)\|_{C([0,T])}\\ &\le C(T,\alpha)\left\{\varepsilon^2+\frac{1}{\varepsilon^2}\exp\left[-\bar{E}(\alpha,T)\left(\frac{h}{\varepsilon}\right)^{\frac{2}{2-\alpha}}\right]+\exp\left[-\bar{C}(\alpha,T)\frac{h}{\varepsilon}\right]\right\}\\ &\le C(T,\alpha)\left\{h^{2+2\gamma_1}+\frac{1}{h^{2+2\gamma_1}}\exp\left[-\bar{E}(\alpha,T)h^{\frac{2\gamma_1}{\alpha-2}}\right]+\exp\left[-\frac{\bar{C}(\alpha,T)}{h^{\gamma_1}}\right]\right\},\end{aligned} \tag{2.33}$$

where the constants $C(T,\alpha),\bar{E}(\alpha,T),\bar{C}(\alpha,T)$ are independent of x,t,ε. Besides, as $0<h\ll\varepsilon$, the following estimation holds:

$$\max_{j=0,\ldots,2N}\|e_j(t)\|_{C([0,T])}\le C(T,\alpha)\left(h+\frac{h}{\varepsilon}+\frac{h^2}{\varepsilon^2}+\frac{h^2}{\varepsilon^4}+\frac{h^2}{\varepsilon^6}\right) \tag{2.34}$$

with the constant $C(T,\alpha)$ independent of x,t,ε.

Proof. Let us start with the case of $\varepsilon\ll h$. We can decompose the truncation error $\mathbb{T}_h u_j(t)$ into two parts

$$\mathbb{T}_h u_j(t)=\mathbb{T}_{h,1}u_j(t)-\mathbb{T}_{h,2}u_j(t),$$

where we denote

$$\mathbb{T}_{h,1}u_j(t)={}_0^C D_t^\alpha u_j(t)-f_j(t), \tag{2.35}$$

$$\mathbb{T}_{h,2}u_j(t)=\frac{\varepsilon^2}{\bar{h}_j}\left(a_{j+\frac{1}{2}}\bar{\delta}_x u_{j+\frac{1}{2}}-a_{j-\frac{1}{2}}\bar{\delta}_x u_{j-\frac{1}{2}}\right)(t). \tag{2.36}$$

For the first term $\mathbb{T}_{h,1}u_j(t)$, it follows from the results in Proposition 2.2 and the following facts:

$$h-1\le x_j\le 1-h,\quad 0<t\le T,$$

to obtain

$$\begin{aligned}|\mathbb{T}_{h,2}u(x_j,t)|&=\left|\varepsilon^2\partial_x\left(a(x_j)\partial_x\right)u(x_j,t)\right|\\ &\le\left|\varepsilon^2 a(x_j)\partial_{xx}u(x_j,t)\right|+\left|\varepsilon^2 a'(x_j)\partial_x u(x_j,t)\right|\\ &\le C(T,\alpha)\left[\varepsilon^2+(1+\varepsilon^{-2})e^{-\bar{E}(\alpha)\left[\frac{(1+x_j)^2}{\varepsilon^2 t^\alpha}\right]^{\frac{1}{2-\alpha}}}+(1+\varepsilon^{-2})e^{-\bar{E}(\alpha)\left[\frac{(1-x_j)^2}{\varepsilon^2 t^\alpha}\right]^{\frac{1}{2-\alpha}}}\right]\end{aligned}$$

$$\leq C(T,\alpha)\left[\varepsilon^2+e^{-\bar{E}(\alpha)\left(\frac{h^2}{\varepsilon^2T^\alpha}\right)^{\frac{1}{2-\alpha}}}+\varepsilon^{-2}e^{-\bar{E}(\alpha)\left(\frac{h^2}{\varepsilon^2T^\alpha}\right)^{\frac{1}{2-\alpha}}}\right] \tag{2.37}$$

with some constants $C(T,\alpha),\bar{E}(\alpha)$ independent of x,t and ε.

Next, we estimate the second term $\mathbb{T}_{h,2}u_j(t)$. It follows from the estimation (2.22) in Proposition 2.3 to obtain that the solution $u(x,t)$ for (1.1) is uniformly bounded to ε, that is, there is a constant $C(T,\alpha)$ independent of x,t and ε such that

$$|u(x_j,t)|\leq C(T,\alpha).$$

Besides, as $\varepsilon\ll h$, we have

$$\bar{h}_j\geq\frac{\sqrt{a_j}\,\bar{\varepsilon}}{2}\exp\left[\frac{h}{2\sqrt{a_j}\,\bar{\varepsilon}}\right]\geq C(\alpha)t^{\frac{\alpha}{2}}\varepsilon\exp\left[C(\alpha)ht^{-\frac{\alpha}{2}}\varepsilon^{-1}\right], \tag{2.38}$$

$$\bar{h}_{j+\frac{1}{2}}\geq\frac{\sqrt{a_{j+\frac{1}{2}}}\,\bar{\varepsilon}}{2}\exp\left[\frac{h}{\sqrt{a_{j+\frac{1}{2}}}\,2\bar{\varepsilon}}\right]\geq C(\alpha)t^{\frac{\alpha}{2}}\varepsilon\exp\left[C(\alpha)ht^{-\frac{\alpha}{2}}\varepsilon^{-1}\right] \tag{2.39}$$

with a constant $C(T,\alpha)$ independent of x,t and ε. Hence, we can obtain

$$\begin{aligned}|\mathbb{T}_{h,2}u(x_j,t)|&\leq\left(\frac{\varepsilon^2}{\bar{h}_j\bar{h}_{j+\frac{1}{2}}}+\frac{\varepsilon^2}{\bar{h}_j\bar{h}_{j-\frac{1}{2}}}\right)|u_j(t)|+\frac{\varepsilon^2}{\bar{h}_j\bar{h}_{j-\frac{1}{2}}}|u_{j-1}(t)|+\frac{\varepsilon^2}{\bar{h}_j\bar{h}_{j+\frac{1}{2}}}|u_{j+1}(t)|\\&\leq 2C(T,\alpha)\varepsilon^2\left(\frac{1}{\bar{h}_j\bar{h}_{j+\frac{1}{2}}}+\frac{1}{\bar{h}_j\bar{h}_{j-\frac{1}{2}}}\right)\\&\leq C(T,\alpha)t^{-\frac{\alpha}{2}}\exp\left[-2C(\alpha)\frac{h}{t^{\frac{\alpha}{2}}\varepsilon}\right],\\&\leq C(T,\alpha)\left\{t^{-\frac{\alpha}{2}}\exp\left[-C(\alpha)\frac{h}{t^{\frac{\alpha}{2}}\varepsilon}\right]\right\}\exp\left[-C(\alpha)\frac{h}{T^{\frac{\alpha}{2}}\varepsilon}\right]\\&\leq C(T,\alpha)\exp\left[-C(\alpha)\frac{h}{T^{\frac{\alpha}{2}}\varepsilon}\right],\end{aligned} \tag{2.40}$$

where the constant $C(T,\alpha),C(\alpha)$ are independent of x,t and ε. To sum up the above estimations (2.37) and (2.40), we can obtain the following truncation error estimation:

$$\begin{aligned}|\mathbb{T}_hu_j(t)|&\leq|\mathbb{T}_{h,1}u_j(t)|+|\mathbb{T}_{h,2}u_j(t)|\\&\leq C(T,\alpha)\left\{\varepsilon^2+\frac{1}{\varepsilon^2}\exp\left[-\frac{\bar{E}(\alpha)}{T^{\frac{\alpha}{2-\alpha}}}\left(\frac{h}{\varepsilon}\right)^{\frac{2}{2-\alpha}}\right]+\exp\left[-\frac{C(\alpha)}{T^{\frac{\alpha}{2}}}\frac{h}{\varepsilon}\right]\right\}\\&=C(T,\alpha)\left\{\varepsilon^2+\frac{1}{\varepsilon^2}\exp\left[-\bar{E}(\alpha,T)\left(\frac{h}{\varepsilon}\right)^{\frac{2}{2-\alpha}}\right]+\exp\left[-\bar{C}(\alpha,T)\frac{h}{\varepsilon}\right]\right\}\end{aligned} \tag{2.41}$$

where the constant $C(T,\alpha),\bar{C}(\alpha,T),\bar{E}(\alpha,T)$ are independent of x,t,ε.

In fact, according to the definition of truncation error $\mathbb{T}_h u_j(t)$, the error function $e_j(t)$ satisfies the following differential equations:

$$\begin{cases} {}_0^C D_t^\alpha e_j(t) - \dfrac{\varepsilon^2}{\bar{h}_j}\left[a_{j+\frac{1}{2}}\bar{\delta}_x e_{j+\frac{1}{2}}(t) - a_{j-\frac{1}{2}}\bar{\delta}_x e_{j-\frac{1}{2}}(t)\right] = \mathbb{T}_h u_j(t), & 0<t\le T, \quad |x_j|<1, \\ e_0(t) = e_{2N+1}(t) = 0, & 0<t\le T, \\ e_j(0) = 0, & -1\le x_j \le 1. \end{cases} \tag{2.42}$$

According to the result proposed in Theorem 2.1, the solution $e_j(t)$ satisfies the following estimation:

$$\begin{aligned} &\max_{j=0,\dots,2N} \|e_j(t)\|_{C([0,T])} \\ &\le M_0 + \frac{T^\alpha}{\Gamma(1+\alpha)} M_1 \\ &= 0 + \frac{T^\alpha}{\Gamma(1+\alpha)} \max_{1\le j\le 2N-1} \|\mathbb{T}_h u_j(t)\|_{C([0,T])} \\ &\le C(T,\alpha)\left\{\varepsilon^2 + \frac{1}{\varepsilon^2}\exp\left[-\bar{E}(\alpha,T)\left(\frac{h}{\varepsilon}\right)^{\frac{2}{2-\alpha}}\right] + \exp\left[-\bar{C}(\alpha,T)\frac{h}{\varepsilon}\right]\right\} \end{aligned} \tag{2.43}$$

with some constant $C(T,\alpha), \bar{C}(\alpha,T), \bar{E}(\alpha,T)$ independent of x,t,ε.

Next, we consider the case of $\varepsilon \gg h$. From the Taylor's expansion, we can get

$$\bar{h}_j = h\left[1 + C_j(\alpha)\frac{h^2}{\varepsilon^2}\right], \quad \bar{h}_{j+\frac{1}{2}} = h\left[1 + C_{j+\frac{1}{2}}(\alpha)\frac{h^2}{\varepsilon^2}\right].$$

Hence, we have

$$\left(\bar{h}_j \bar{h}_{j+\frac{1}{2}}\right)^{-1} = h^{-2}\left[1 + \bar{C}_{j+\frac{1}{2}}(\alpha)\frac{h^2}{\varepsilon^2}\right], \quad \left(\bar{h}_j \bar{h}_{j-\frac{1}{2}}\right)^{-1} = h^{-2}\left[1 + \bar{C}_{j-\frac{1}{2}}(\alpha)\frac{h^2}{\varepsilon^2}\right].$$

And then, we can decompose the truncation error $\mathbb{T}_h u_j(t)$ into two parts

$$\mathbb{T}_h u_j(t) = \mathbb{T}_{h,1} u_j(t) - \mathbb{T}_{h,2} u_j(t),$$

where we denote

$$\mathbb{T}_{h,1} u_j(t) = {}_0^C D_t^\alpha u_j(t) - \frac{\varepsilon^2}{h^2}\left[a_{j+\frac{1}{2}}(u_{j+1}-u_j) - a_{j-\frac{1}{2}}(u_j - u_{j-1})\right](t) - f_j(t), \tag{2.44}$$

$$\mathbb{T}_{h,2} u_j(t) = \bar{C}_{j+\frac{1}{2}}(\alpha) a_{j+\frac{1}{2}}(u_{j+1}-u_j)(t) - \bar{C}_{j-\frac{1}{2}}(\alpha) a_{j-\frac{1}{2}}(u_j - u_{j-1})(t). \tag{2.45}$$

For the second part of the above decomposition, the following estimator can be obtained from the mean value theorem:

$$|\mathbb{T}_{h,2} u_j(t)| \le C(\alpha) h\left[\left|\partial_x u\left(\xi_{j+\frac{1}{2}},t\right)\right| + \left|\partial_x u\left(\xi_{j+\frac{1}{2}},t\right)\right|\right]$$

$$
\begin{aligned}
&\leq 2C(\alpha)h\|\partial_x u(\cdot,t)\|_\infty \\
&\leq C_1(\alpha)h + C_2(\alpha)\frac{h}{\varepsilon^2}\exp\left\{-\bar{E}(\alpha)\left[\frac{h^2}{\varepsilon^2 T^\alpha}\right]^{\frac{1}{2-\alpha}}\right\}. \\
&\leq C(\alpha)\left(h+\frac{h}{\varepsilon^2}\right)
\end{aligned} \tag{2.46}
$$

with some constant $C(\alpha)$ independent with h,t and ε. As for the first part $\mathbb{T}_{h,1}u_j(t)$, it follows from the Taylor's expansion and the mean value theorem that there are some constants $\xi_k \in [x_{i-1},x_{i+1}], k=1,2,3,4$ such that

$$
\begin{aligned}
\mathbb{T}_{h,1}u_j(t) = &\frac{\varepsilon^2 h^2}{24}a^{(3)}(\xi_1)\partial_x u(x_j,t) + \frac{\varepsilon^2 h^2}{8}a^{(2)}(\xi_2)\partial_x^{(2)}u(x_j,t) \\
&+\frac{\varepsilon^2 h^2}{6}a'(\xi_3)\partial_x^{(3)}u(x_j,t) + \frac{\varepsilon^2 h^2}{12}a_j\partial_x^{(3)}u(\xi_4,t).
\end{aligned}
$$

Then, from the estimation (2.6), we can get

$$
|\mathbb{T}_{h,1}u_j(t)| \leq C(T,\alpha)\left(h^2+\frac{h^2}{\varepsilon^2}+\frac{h^2}{\varepsilon^4}+\frac{h^2}{\varepsilon^6}\right) \tag{2.47}
$$

with some constant $C(\alpha)$ independent with h,t and ε. Therefore, for truncation error $\mathbb{T}_h u_j(t)$, we have the following estimation:

$$
|\mathbb{T}_h u_j(t)| \leq C(T,\alpha)\left(h+\frac{h}{\varepsilon}+\frac{h^2}{\varepsilon^2}+\frac{h^2}{\varepsilon^4}+\frac{h^2}{\varepsilon^6}\right) \tag{2.48}
$$

with some constant $C(\alpha)$ independent with h,t and ε. So, we have as $0<h\ll\varepsilon$,

$$
\max_{j=0,\dots,2N}\|e_j(t)\|_{C([0,T])} \leq C(T,\alpha)\left(h+\frac{h}{\varepsilon}+\frac{h^2}{\varepsilon^2}+\frac{h^2}{\varepsilon^4}+\frac{h^2}{\varepsilon^6}\right) \tag{2.49}
$$

with some constant $C(\alpha)$ independent with h,t and ε. □

Remark 2.5. From the error estimation (2.33), we know that for a given h, the approximate solution $\{U_j(t)\}$ obtained by the semi-discrete TFPM scheme (2.19) becomes more and more accurate as $\gamma_1 \to +\infty$ and $\varepsilon \to 0^+$. For a given ε, it can be seen from the estimation (2.34) that as $h \to 0^+$, the numerical solution $U_j(t)$ converts to the analytical solution $u(x_j,t)$.

Remark 2.6. In the estimation (2.33), as $\gamma_1 \to 0^+$, the exponential items will bring relatively large errors because of the resonance effect, which will affect the numerical precision of the newly constructed TFPM scheme. How to eliminate the resonance effect at $h\sim\varepsilon$ will be the focus of our future research.

2.4 The $\mathcal{L}1$-TFPM scheme

In generally, the analytical solutions of ordinary differential equations (2.20) cannot be written. Therefore, we need to solve the ordinary differential equations (2.20) numerically.

Since the solution $u(x,t)$ of the subdiffusion equation will have a weak singularity at the initial time $t=0$, we will use the following graded grid proposed in [19] in the time direction

$$t_k = T(k/M)^{\gamma}. \tag{2.50}$$

As $\gamma=1$, we can get a uniform grid with step size $\tau=1/M$, as $\gamma>1$, the graded grid becomes finer near $t=0$ to capture the singularity at the initial time.

We exploit the $\mathcal{L}1$ formula proposed in [20] to approximate the Caputo fractional derivative ${}_0^C D_t^{\alpha}\vec{\mathbf{U}}(t)$ in (2.20), that is,

$${}_0^C D_t^{\alpha}\vec{\mathbf{U}}(t) \approx \sum_{k=1}^{n} \frac{a_{n-k}^{(\alpha)}}{\Gamma(2-\alpha)} \Delta_\tau \vec{\mathbf{U}}\left(t^{k-\frac{1}{2}}\right), \tag{2.51}$$

where the coefficients $a_{n-k}^{(\alpha)}$ are defined by

$$a_{n-k}^{(\alpha)} = \frac{(t^n - t^{k-1})^{1-\alpha} - (t^n - t^k)^{1-\alpha}}{\tau_k}, \quad k=0,1,2,\ldots,n-1$$

with the time step τ_k defined as $\tau_k = t^k - t^{k-1}$, and we denote

$$\Delta_\tau w^{k-\frac{1}{2}} = w^k - w^{k-1}.$$

If we set

$$D_\tau^{\alpha} w^n = \sum_{k=1}^{n} \frac{a_{n-k}^{(\alpha)}}{\Gamma(2-\alpha)} \Delta_\tau^{k-\frac{1}{2}} w,$$

$$\vec{\mathbf{U}}^n = \left[U_1^n, U_2^n, \cdots, U_{2N-2}^n, U_{2N-1}^n\right]^{\top},$$

the initial value problem (2.20) can be discretized as follows:

$$\begin{cases} D_\tau^{\alpha}\vec{\mathbf{U}}^n + \mathbf{A}_h^{\varepsilon}(t^n)\vec{\mathbf{U}}^n = \vec{\mathbf{F}}(t^n), & 0<t^n\le T, \\ \vec{\mathbf{U}}^0 = [w(x_1), w(x_2), \cdots, w(x_{2N-2}), w(x_{2N-1})]^{\top}. \end{cases} \tag{2.52}$$

Furthermore, we give a new $\mathcal{L}1$-TFPM scheme to solve the initial-boundary value problem (1.1)

$$\begin{cases} D_\tau^{\alpha} U_j^n - \dfrac{\varepsilon^2}{\bar{h}_j}\left(a_{j+\frac{1}{2}}\bar{\delta}_x U_{j+\frac{1}{2}}^n - a_{j-\frac{1}{2}}\bar{\delta}_x U_{j-\frac{1}{2}}^n\right) = f_j^n, & (x_j,t^n)\in\Omega_{h,\tau}^T, \\ U_0^n = \phi(t^n), \quad U_{2N+1}^n = \psi(t^n), & 0\le t^n\le T, \\ U_j^0 = w(x_j), & -1\le x_j\le 1, \end{cases} \tag{2.53}$$

where the notations $D_\tau^{\alpha} U_j^n, \bar{\delta}_x U_{j\pm1/2}^n$ and $\bar{h}_j$ are defined as before.

Remark 2.7. Similar to the discussion in Remark 2.1, we can know that as $h \ll \varepsilon$, the newly proposed $\mathcal{L}$1-TFPM scheme will degenerate to the traditional $\mathcal{L}$1-FDM scheme,

$$D_\tau^\alpha U_j^n - \frac{\varepsilon^2}{h^2}\left[a_{j+\frac{1}{2}}\Delta_x U_{j+\frac{1}{2}}^n - a_{j-\frac{1}{2}}\Delta_x U_{j-\frac{1}{2}}^n\right] = f_j^n, \quad t^n > 0, \tag{2.54}$$

so it is an unconditionally consistent numerical scheme.

Next, we show that the newly proposed $\mathcal{L}$1-TFPM scheme (2.53) satisfies the discrete extremum principle. Before that, we need the following lemma.

Lemma 2.2. *Let the grid function $\{v^k\}_{k=1}^M$ be defined on the grid $\{t^k\}_{k=1}^M$. If v^n is the maximum point of $\{v^k\}_{k=1}^M$, then we have*

$$D_\tau^\alpha v^n \geq 0. \tag{2.55}$$

Proof. We can write $D_\tau^\alpha v^n$ as

$$\begin{aligned} D_\tau^\alpha v^n &= \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^n b_k (v^n - v^{k-1}) \\ &= \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^n b_k \sum_{l=k}^n \Delta_\tau v^{l-\frac{1}{2}} \\ &= \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^n \left(\sum_{l=1}^k b_l\right)\Delta_\tau v^{k-\frac{1}{2}}. \end{aligned}$$

According to the definition of $D_\tau^\alpha v^n$, it is easy to see that

$$\sum_{l=1}^k b_l = a_{n-k}^{(\alpha)}, \quad k = 1,2,\ldots,n.$$

If we set $a_n^{(\alpha)} = 0$, we can obtain

$$b_k = a_{n-k}^{(\alpha)} - a_{n-k+1}^{(\alpha)}, \quad k = 1,2,\ldots,n.$$

It follows from Lagrange's mean value theorem that there exists a parameter $0 < \theta_k < 1$ such that

$$\alpha_{n-k}^{(\alpha)} = \frac{1-\alpha}{\left[t^n - \theta_k(t^k) - (1-\theta_k)t^{k-1}\right]^\alpha},$$

According to the monotonicity of the function $H(t) = (1-\alpha)t^{-\alpha}$, we can obtain that

$$b_k > 0, \quad k = 1,2,\ldots,n.$$

If v^n is the maximum point of $\{v^k\}_{k=1}^M$, and then we can obtain

$$v^n - v^{k-1} \geq 0, \quad k = 1,2,\ldots,n.$$

Furthermore, we can get

$$D_\tau^\alpha v^n = \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{n} b_k (v^n - v^{k-1}) \geq 0.$$

The proof is complete. □

The discrete maximum principle for the $\mathcal{L}1$-TFPM scheme (2.53) is given by the following proposition.The proof of the following proposition is based on the idea for the traditional $\mathcal{L}1$-FDM proposed in [1].

Proposition 2.6 (Discrete Maximum Principle)**.** *Let a grid function* $\{U_j^n, (x_j,t^n) \in \Omega_{h,\tau}^T\}$ *be a solution of the* $\mathcal{L}1$*-TFPM scheme* (2.53)*. If the force term* $f_j^n \leq 0$*, and then either* $U_j^n \leq 0$ *or the function* $\{U_j^n\}$ *attains its positive maximum on the bottom or back-side parts*

$$S_{h,\tau}^T = \{(x_j,t^0)\}_{j=0}^{2N} \bigcup \{(x_0,t^n)\}_{n=0}^{TM} \bigcup \{(x_{2N},t^n)\}_{n=0}^{TM},$$

i.e.

$$U_j^n \leq \max\left\{0, \max_{(x_j,t^n)\in S_{h,\tau}^T} U_j^n\right\}, \quad \forall (x_j,t^n) \in \Omega_{h,\tau}^T. \tag{2.56}$$

Proof. We prove this theorem by contradiction. We first suppose that the statement of the theorem does not hold true, that is, there exists a point $(x_{j_0},t^{n_0}), -1 < x_{j_0} < 1, 0 < t^{n_0} \leq T$ such that

$$U_{j_0}^{n_0} > \mathcal{M} = \max\left\{0, \max_{(x_j,t^n)\in S_{h,\tau}^T} U_j^n\right\} > 0.$$

Let us introduce a number $\Delta = U_{j_0}^{n_0} - \mathcal{M}$ and define an auxiliary grid function

$$W_j^n = U_j^n + \frac{\Delta}{2}\frac{T-t^n}{T}, \quad (x_j,t^n) \in \Omega_{h,\tau}^T.$$

The following results can be easily obtained

$$W_j^n \leq U_j^n + \frac{\Delta}{2}, \qquad (x_j,t^n) \in \Omega_{h,\tau}^T, \tag{2.57}$$

$$W_{j_0}^{n_0} \geq U_{j_0}^{n_0} = \Delta + \mathcal{M} \geq \Delta + U_j^n \geq \Delta + W_j^n - \frac{\Delta}{2} \geq \frac{\Delta}{2} + W_j^n, \quad (x_j,t^n) \in S_{h,\tau}^T. \tag{2.58}$$

From the inequality (2.58), we can know that W_j^n cannot reach a maximum on $S_{h,\tau}^T$. If the maximum point of the grid function $\{W_j^n\}$ over the grid $\Omega_{h,\tau}^T$ is denoted by $W_{j_1}^{t_1}$ with $-1 < x_{j_1} < 1, 0 < t^{n_1} \leq T$ and

$$W_{j_1}^{t_1} \geq W_{j_0}^{n_0} \geq \Delta + \mathcal{M} > \Delta.$$

According to Lemma 2.2, we know that

$$D_\tau^\alpha W_{j_1}^{n_1} \geq 0. \tag{2.59}$$

Furthermore, we can easily get

$$\begin{aligned}
&-\left(a_{j_1+\frac{1}{2}}\bar{\delta}_x W_{j_1+\frac{1}{2}}^{n_1} - a_{j_1-\frac{1}{2}}\bar{\delta}_x W_{j_1-\frac{1}{2}}^{n_1}\right)\\
=&a_{j_1-\frac{1}{2}}\bar{\delta}_x W_{j_1-\frac{1}{2}}^{n_1} - a_{j_1+\frac{1}{2}}\bar{\delta}_x W_{j_1+\frac{1}{2}}^{n_1}\\
\geq&\frac{\bar{a}_1}{\bar{h}}\left(2W_{j_1}^{n_1} - W_{j_1+1}^{n_1} - W_{j_1-1}^{n_1}\right) \geq 0.
\end{aligned}$$

According to the definition of W_j^n, we have

$$U_j^n = W_j^n - \frac{\Delta}{2}\frac{T-t^n}{T}.$$

By simple calculation, we can get

$$D_\tau^\alpha (t^n - T) = \frac{(t^n)^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Furthermore, we can get the following relation:

$$D_\tau^\alpha U_j^n = D_\tau^\alpha W_j^n + \frac{\Delta}{2T}\frac{(t^n)^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{2.60}$$

Additionally, we have

$$-\left(a_{j_1+\frac{1}{2}}\bar{\delta}_x U_{j_1+\frac{1}{2}}^{n_1} - a_{j_1-\frac{1}{2}}\bar{\delta}_x U_{j_1-\frac{1}{2}}^{n_1}\right) = -\left(a_{j_1+\frac{1}{2}}\bar{\delta}_x W_{j_1+\frac{1}{2}}^{n_1} - a_{j_1-\frac{1}{2}}\bar{\delta}_x W_{j_1-\frac{1}{2}}^{n_1}\right) \geq 0. \tag{2.61}$$

Combining (2.59)-(2.61), we can obtain

$$L_{h,\tau}^\varepsilon U_{j_1}^{n_1} = D_\tau^\alpha W_{j_1}^{n_1} + \frac{\Delta}{2T}\frac{(t^{n_1})^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\varepsilon^2}{\bar{h}_{j_1}}\left(a_{j_1+\frac{1}{2}}\bar{\delta}_x W_{j_1+\frac{1}{2}}^{n_1} - a_{j_1-\frac{1}{2}}\bar{\delta}_x W_{j_1-\frac{1}{2}}^{n_1}\right) > 0, \tag{2.62}$$

where the discrete operator $L_{h,\tau}^\varepsilon$ defined by

$$L_{h,\tau}^\varepsilon U_j^n = D_\tau^\alpha U_j^n - \frac{\varepsilon^2}{\bar{h}_j}\left(a_{j+\frac{1}{2}}\bar{\delta}_x U_{j+\frac{1}{2}}^n - a_{j-\frac{1}{2}}\bar{\delta}_x U_{j-\frac{1}{2}}^n\right).$$

On the other hand, by the conditions of the theorem, we have

$$L_{h,\tau}^\varepsilon U_{j_1}^{n_1} = f_{j_1}^{n_1} \leq 0,$$

which contradicts the inequality (2.62). Above all, the assumption made at the beginning of the theorem proof is wrong and thus this theorem is proved. □

Substituting $-U_j^n$ instead of in the reasoning above, the discrete minimum principle can be obtained.

Proposition 2.7 (Discrete Minimum Principle). *Let a grid function* $\{U_j^n,(x_j,t^n)\in\Omega_{h,\tau}^T\}$ *be a solution of the* $\mathcal{L}1$*-TFPM scheme* (2.53)*. If the force term* $f_j^n\geq 0$*, and then either* $U_j^n\geq 0$ *or the function* $\{U_j^n\}$ *attains its negative minimum on the bottom or back-side parts*

$$S_{h,\tau}^T=\left\{(x_j,t^0)\right\}_{j=0}^{2N}\bigcup\left\{(x_0,t^n)\right\}_{n=0}^{TM}\bigcup\left\{(x_{2N},t^n)\right\}_{n=0}^{TM},$$

i.e.

$$U_j^n\geq\min_{(x_j,t^n)\in S_{h,\tau}^T}U_j^n,\quad\forall(x_j,t^n)\in\Omega_{h,\tau}^T.\tag{2.63}$$

Based on the discrete extremum principle above, we can prove that the newly proposed $\mathcal{L}1$-TFPM scheme (2.53) is unconditionally stable.

Theorem 2.3 (The Stability of the $\mathcal{L}1$-TFPM Scheme). *Let a grid function* $\{U_j^n,(x_j,t^n)\in\Omega_{h,\tau}^T\}$ *be a solution of the* $\mathcal{L}1$*-TFPM scheme* (2.53)*. If we denote*

$$\begin{aligned}&M_0=\max\{\|w(x)\|_{C([-1,1])},\|\phi(t)\|_{C([0,T])},\|\psi(t)\|_{C([0,T])}\},\\&M_1=\|f(x,t)\|_{C(\Omega)},\end{aligned}$$

and then the following estimate of the solution norm holds true:

$$\|U_j^n\|_\infty\leq M_0+\frac{T^\alpha}{\Gamma(1+\alpha)}M_1.\tag{2.64}$$

Proof. To prove the theorem, we first introduce an auxiliary grid function $\{W_j^n\}$,

$$W_j^n=U_j^n-\frac{M_1}{\Gamma(1+\alpha)}(t^n)^\alpha.$$

According to the analysis in [14], if we denote

$$\omega_{1+\alpha}(t)=\frac{t^\alpha}{\Gamma(1+\alpha)},$$

we have that

$${}_0^CD_t^\alpha\omega_{1+\alpha}(t^n)-D_\tau^\alpha\omega_{1+\alpha}^n=\sum_{k=1}^n\int_{t^{k-1}}^{t^k}\omega''_{1+\alpha}(s)\tilde{\Pi}_k\omega_{2-\alpha}(t^n-s)ds,$$

where the operator $\tilde{\Pi}_k$ is an interpolation operator with

$$\tilde{\Pi}_k\omega_{2-\alpha}(t^n-s)\geq 0.$$

Besides, we can obtain

$$\omega''_{1+\alpha}(t) = -\frac{1-\alpha}{\Gamma(\alpha)t^{2-\alpha}} < 0, \quad t > 0.$$

Furthermore, we can get

$$D_\tau^\alpha \omega_{1+\alpha}^n \geq {}_0^C D_t^\alpha \omega_{1+\alpha}(t^n) = 1.$$

Then, we know

$$L_{h,\tau}^\varepsilon W_j^n = L_{h,\tau}^\varepsilon U_j^n - M_1 D_\tau^\alpha \omega_{1+\alpha}^n \leq f_j^n - M_1 \leq 0.$$

Then, according to Proposition 2.6, we can get that for any $(x_j,t^n) \in \Omega_{h,\tau}^T$,

$$\begin{aligned} W_j^n \leq \max\left\{0, \max_{(x_j,t^n)\in S_{h,\tau}^T} W_j^n\right\} &\leq \max\left\{0, \max_{(x_j,t^n)\in S_{h,\tau}^T} U_j^n - \frac{M_1}{\Gamma(1+\alpha)}(t^n)^\alpha\right\} \\ &\leq \max\left\{0, \max_{(x_j,t^n)\in S_{h,\tau}^T} U_j^n\right\} \leq M_0. \end{aligned}$$

Furthermore, we can obtain for any $(x_j,t^n) \in \Omega_{h,\tau}^T$,

$$U_j^n \leq M_0 + \frac{M_1}{\Gamma(1+\alpha)}(t^n)^\alpha \leq M_0 + \frac{M_1}{\Gamma(1+\alpha)} T^\alpha.$$

Besides, we first introduce another auxiliary grid function $\{V_j^n\}$,

$$V_j^n = U_j^n + \frac{M_1}{\Gamma(1+\alpha)}(t^n)^\alpha.$$

It follows from a simple calculation to obtain

$$L_{h,\tau}^\varepsilon V_j^n = L_{h,\tau}^\varepsilon U_j^n + M_1 D_\tau^\alpha \omega_{1+\alpha}^n \geq f_j^n + M_1 \geq 0.$$

Then, according to Proposition 2.7, we can get that for any $(x_j,t^n) \in \Omega_{h,\tau}^T$,

$$\begin{aligned} V_j^n &\geq \min_{(x_j,t^n)\in S_{h,\tau}^T} V_j^n \geq \min_{(x_j,t^n)\in S_{h,\tau}^T} \left\{U_j^n + \frac{M_1}{\Gamma(1+\alpha)}(t^n)^\alpha\right\} \\ &\geq \min_{(x_j,t^n)\in S_{h,\tau}^T} U_j^n \geq -M_0. \end{aligned}$$

Furthermore, we can obtain for any $(x_j,t^n) \in \Omega_{h,\tau}^T$,

$$U_j^n \geq -M_0 - \frac{M_1}{\Gamma(1+\alpha)}(t^n)^\alpha \geq M_0 - \frac{M_1}{\Gamma(1+\alpha)} T^\alpha.$$

In summary, the content of the theorem is proved. □

Remark 2.8. From the stability of the $\mathcal{L}$1-TFPM scheme (2.53), we can know that the numerical solution obtained by the numerical scheme (2.53) is unique and continuously depends on the initial-boundary values $\phi(t), \psi(t), w(x)$ and the source term $f(x,t)$.

3 Numerical experiments

In this section, several numerical examples are given to demonstrate the validity of the newly proposed $\mathcal{L}$1-TFPM scheme (2.53). We consider an initial-boundary value problem on $\Omega = [-1,1]\times[0,1]$ as follow:

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) - \varepsilon^2 \partial_x\left[(x^2+1)\partial_x\right] u(x,t) = 0.3e^{\frac{1}{1+t}}(1-x^2), & (x,t)\in\Omega, \\ u(-1,t)=t^2, \quad u(1,t)=t^2, & 0\le t\le 1, \\ u(x,0)=e^{\frac{1}{x^2-1}}, & x\in[-1,1]. \end{cases} \tag{3.1}$$

We use the $\mathcal{L}$1-TFPM scheme (2.53) to solve the above initial-boundary value problem.

We first examine whether the newly proposed $\mathcal{L}$1-TFPM scheme (2.53) can satisfy the discrete extremum principle. The source term $f(x,t)=t(x^2+2)$ of (3.1) is non-negative and both the initial and boundary values of (3.1) are non-negative. According to the extremum principle, the solution $u(x,t)$ of the initial-boundary value problem (3.1) is greater than zero in $\Omega^0=(-1,1)\times(0,T]$. Fig. 1 shows the numerical solution computed from the $\mathcal{L}$1-TFPM scheme of (3.1) for different α and ε on a uniform grid with $h,\tau=0.01$. It can be seen that for different ε and α, the numerical solutions $U_{h,\tau}$ are all greater than 0 on the internal grid. Thus, the numerical solution computed from the $\mathcal{L}$1-TFPM scheme (2.53) satisfies the discrete extremum value principle.

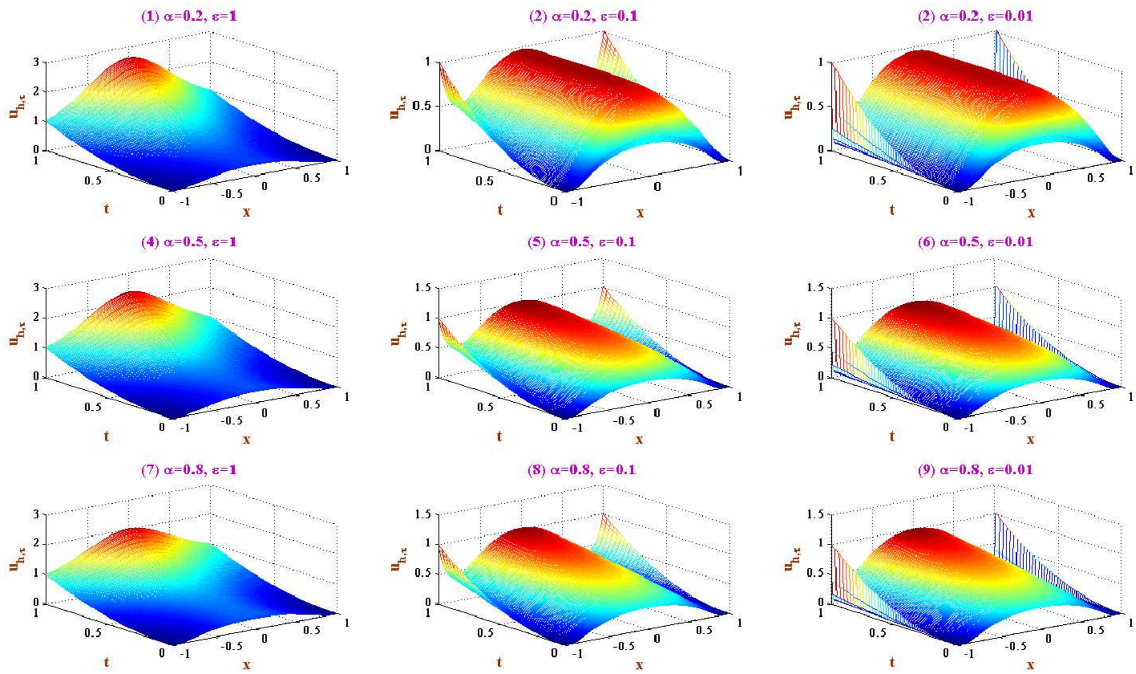

Figure 1: The numerical solution of the initial-boundary value problem (3.1) for different α and ε as $h,\tau=0.01$.

Next, we examine the numerical precision of the $\mathcal{L}1$-TFPM scheme. Since the analytical solution $u(x,t)$ for the initial-boundary value problem (3.1) cannot be given, the numerical solution on a uniform grid with $h=1/1600, \tau=0.0005$ is selected as the reference solution $U^n_{exact,j}$. And then we use the L^∞ error norm $E^n_{h,\tau}$ defined below to examine the numerical error of the numerical solution

$$E^n_{h,\tau} = \max_{0\le j\le 2N+1} \left| U^n_j - U^n_{exact,j} \right|.$$

We compare the newly proposed $\mathcal{L}1$-TFPM scheme with the traditional $\mathcal{L}1$-FDM scheme and the $\mathcal{L}1$-TFPM scheme given in [12]. For convenience, we will label the newly proposed $\mathcal{L}1$-TFPM scheme as $\mathcal{L}1$-TFPM-I scheme, and then label the $\mathcal{L}1$-TFPM scheme given in [12] as $\mathcal{L}1$-TFPM-II scheme.

Fig. 2 shows the numerical precision for the $\mathcal{L}1$-TFPM-I scheme as $h \ll \varepsilon$. The step size we chose for the time direction is a uniform grid with $\tau = 0.0005$. It can be seen that for different ε, the L^∞ error $E^M_{h,\tau}$ decays with a first-order velocity when $h \to 0^+$, which is consistent with the convergence rate given by the estimation (2.34).

Tables 1-3 show the L^∞ errors of the numerical solution for (3.1) with different ε and α as $\tau = 0.0005$ and $h = 0.05$. It can be seen that as $\varepsilon \to 0^+$, the numerical error of the numerical solution obtained by the $\mathcal{L}1$-TFPM-I scheme approaches zero, which is consistent with the error estimation (2.33) in Theorem 2.1. As $h \sim \varepsilon\tau^{\alpha/2}$, the numerical precisions of the $\mathcal{L}1$-TFPM-I scheme and the $\mathcal{L}1$-FDM scheme are close, and both are higher than the numerical precision of the $\mathcal{L}1$-TFPM-II scheme. In the case of $0<\varepsilon \ll h$, the numerical precisions of the $\mathcal{L}1$-TFPM-I scheme and the $\mathcal{L}1$-TFPM-II scheme are similar and much higher than that of the $\mathcal{L}1$-FDM scheme.

Tables 4-6 show the L^∞ errors of the numerical solution for (3.1) with different spatial step h and α as $\tau = 0.0005$. When $h \gg \varepsilon$, both of the numerical solutions obtained by the

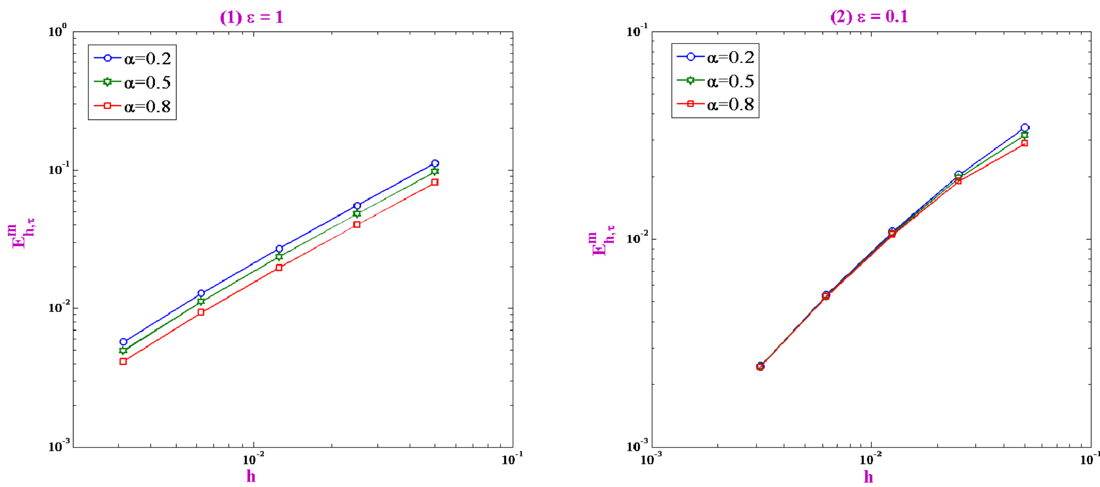

Figure 2: The numerical precision for the $\mathcal{L}1$-TFPM-I scheme as $h \ll \varepsilon$: (1) $\varepsilon=1$; (2) $\varepsilon=0.1$.

Table 1: L^∞ errors of the numerical solution for (3.1) with different ε as $\alpha=0.2, h=0.05$ and $\tau=0.0005$.

ε	$\mathcal{L}$1-TFPM-I scheme	$\mathcal{L}$1-TFPM-II scheme	$\mathcal{L}$1-FDM scheme
0.1	3.4528×10^{-2}	4.0608×10^{-2}	3.3228×10^{-2}
0.01	5.3799×10^{-3}	1.8642×10^{-2}	3.7036×10^{-2}
0.001	8.9405×10^{-6}	8.9405×10^{-6}	6.4459×10^{-4}
0.0001	3.3023×10^{-8}	3.3023×10^{-8}	6.4210×10^{-6}

Table 2: L^∞ errors of the numerical solution for (3.1) with different ε as $\alpha=0.5, h=0.05$ and $\tau=0.0005$.

ε	$\mathcal{L}$1-TFPM-I scheme	$\mathcal{L}$1-TFPM-II scheme	$\mathcal{L}$1-FDM scheme
0.1	3.1803×10^{-2}	1.1670×10^{-1}	3.0844×10^{-2}
0.01	1.4503×10^{-2}	9.1132×10^{-3}	3.3846×10^{-2}
0.001	8.9571×10^{-6}	8.9571×10^{-6}	4.7084×10^{-4}
0.0001	3.8543×10^{-8}	3.8543×10^{-8}	4.6884×10^{-6}

Table 3: L^∞ errors of the numerical solution for (3.1) with different ε as $\alpha=0.8, h=0.05$ and $\tau=0.0005$.

ε	$\mathcal{L}$1-TFPM-I scheme	$\mathcal{L}$1-TFPM-II scheme	$\mathcal{L}$1-FDM scheme
0.1	2.8829×10^{-2}	5.6539×10^{-1}	2.8382×10^{-2}
0.01	2.1181×10^{-2}	2.4831×10^{-3}	2.8873×10^{-2}
0.001	7.6790×10^{-6}	7.6790×10^{-6}	3.3392×10^{-4}
0.0001	4.8452×10^{-8}	4.8452×10^{-8}	3.3362×10^{-6}

Table 4: L^∞ errors of the numerical solution for (3.1) with different spatial step h as $\alpha=0.2, \varepsilon=0.001$ and $\tau=0.0005$.

Spatial step h	$\mathcal{L}$1-TFPM-I scheme	$\mathcal{L}$1-TFPM-II scheme	$\mathcal{L}$1-FDM scheme
1/10	8.9001×10^{-6}	8.9001×10^{-6}	1.5197×10^{-4}
1/20	8.9405×10^{-6}	8.9405×10^{-6}	6.4459×10^{-4}
1/40	9.4109×10^{-6}	9.3687×10^{-6}	2.5944×10^{-3}
1/80	1.3711×10^{-4}	4.3844×10^{-4}	1.0232×10^{-2}
1/160	3.9536×10^{-3}	7.4460×10^{-3}	3.1271×10^{-2}
1/320	9.0141×10^{-3}	5.9844×10^{-2}	3.9864×10^{-2}

Table 5: L^∞ errors of the numerical solution for (3.1) with different spatial step h as $\alpha=0.5, \varepsilon=0.001$ and $\tau=0.0005$.

Spatial step h	$\mathcal{L}$1-TFPM-I scheme	$\mathcal{L}$1-TFPM-II scheme	$\mathcal{L}$1-FDM scheme
1/10	8.9571×10^{-6}	8.9571×10^{-6}	1.0933×10^{-4}
1/20	8.9571×10^{-6}	8.9571×10^{-6}	4.7084×10^{-4}
1/40	9.4609×10^{-6}	9.1500×10^{-6}	1.8954×10^{-3}
1/80	3.6310×10^{-4}	9.1500×10^{-6}	7.5452×10^{-3}
1/160	8.9952×10^{-3}	2.9487×10^{-3}	2.6133×10^{-2}
1/320	2.2670×10^{-2}	5.9480×10^{-2}	4.1473×10^{-2}

Table 6: L^∞ errors of the numerical solution for (3.1) with different spatial step h as $\alpha=0.8,\varepsilon=0.001$ and $\tau=0.0005$.

Spatial step h	$\mathcal{L}1$-TFPM scheme 1	$\mathcal{L}1$-TFPM scheme 2	$\mathcal{L}1$-FDM scheme
1/10	7.6790×10^{-6}	7.6790×10^{-6}	7.6152×10^{-5}
1/20	7.6790×10^{-6}	7.6790×10^{-6}	3.3392×10^{-4}
1/40	1.9843×10^{-5}	7.7653×10^{-6}	1.3438×10^{-3}
1/80	1.1916×10^{-3}	7.7653×10^{-6}	5.3727×10^{-3}
1/160	1.3215×10^{-2}	6.1004×10^{-4}	2.0384×10^{-2}
1/320	3.5114×10^{-2}	3.4073×10^{-2}	4.2229×10^{-2}

$\mathcal{L}1$-TFPM-I scheme and the $\mathcal{L}1$-TFPM-II scheme are more accurate than that obtained by $\mathcal{L}1$-FDM scheme. However, in the case of $h\to\varepsilon$, the precision of numerical solutions given by these three numerical schemes decreases due to the resonance effect, especially the $\mathcal{L}1$-TFPM-II scheme. However, in this case, the accuracy of our new $\mathcal{L}1$-TFPM scheme is still higher than that of the traditional $\mathcal{L}1$-FDM scheme for different α. It can be seen from the numerical results that the smaller α is, the more obvious the resonance effect will be.

In addition, we also investigate the numerical accuracy of the time direction for the $\mathcal{L}1$-TFPM-I scheme. Fig. 1 shows that the solution $u(x,t)$ of (3.1) has a weak singularity at the initial time t = 0. We use the $\mathcal{L}1$-TFPM-I scheme to solve (3.1) on graded grids as $\varepsilon=0.001$ and $h=1/1600$. We still use the L^∞ error norm $E^n_{h,\tau}$ defined above to investigate the numerical error. Fig. 3 shows the numerical accuracy in time direction for different α

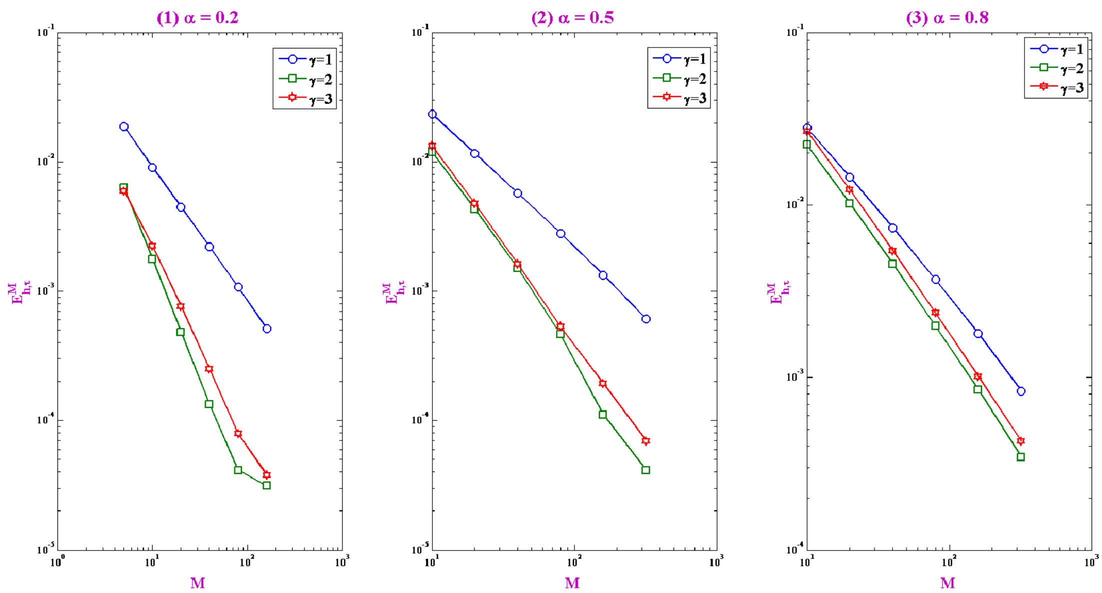

Figure 3: The numerical accuracy in time direction for different α as $\varepsilon=0.001$ and $h=1/1600$.

and γ. The $\mathcal{L}1$-TFPM-I scheme only has $\mathcal{O}(\tau^\alpha)$ precision on the uniform grid due to the non-smooth initial value. However, the numerical accuracy can be greatly improved by using a non-uniform grid with $\gamma > 1$ in the direction of time.

To sum up, the newly proposed $\mathcal{L}1$-TFPM scheme can maintain the discrete extremum principle and has high numerical accuracy as $h \gg \varepsilon$. However, as $h \sim \varepsilon$, the resonance effect results in a large loss of numerical accuracy. How to eliminate the effect of resonance will be the focus of our future research.

4 Conclusion

For the singularly perturbed subdiffusion equations on a bounded domain Ω_B^T, the solution $u(x;t)$ has a boundary layer of width $\mathcal{O}(\varepsilon)$ near the boundary imposed non-smooth boundary values, which presents great challenges for the construction of efficient numerical scheme. In order to construct a high-precision numerical scheme for the singularly perturbed subdiffusion equations on rough grids, we construct a new scheme based on the idea of tailored finite point method. The newly proposed $\mathcal{L}1$-TFPM scheme can preserve the discrete extremum principle and has a higher numerical accuracy than the traditional $\mathcal{L}1$-FDM scheme as $h \gg \varepsilon$. But when $\varepsilon \sim h$, the resonance effect will reduce the accuracy of the newly proposed scheme, which needs our further study.

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