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Error Estimates of Finite Element Methods for the Nonlinear Backward Stochastic Stokes Equations

Yongwang Sun, Weidong Zhao and Wenju Zhao*

School of Mathematics, Shandong University, Jinan 250100, China.

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Abstract. This paper is concerned with the numerical analyses of finite element methods for the nonlinear backward stochastic Stokes equations (BSSEs) where the forcing term is coupled with z. Under several developed analysis techniques, the error estimates of the proposed semi-discrete and fully discrete schemes, as well as their boundedness, are rigorously presented and established. Optimal convergence rates of the fully discrete scheme are obtained not only for the velocity u and auxiliary stochastic process z but also for the pressure p. For the efficiency of solving BSSEs, the proposed numerical scheme is parallelly designed in stochastic space. Numerical results are finally provided and tested in parallel to validate the theoretical results.

AMS subject classifications: 60H35, 65M60, 65M15, 76D07

Key words: Backward stochastic Stokes equations, variational methods, finite element method, error estimates.

1 Introduction

We consider a class of nonlinear backward stochastic Stokes equations defined on a complete and filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$

$$\begin{cases} -\mathrm{d}u_t-\nu\Delta u_t\mathrm{d}t+\nabla p_t\mathrm{d}t=f(t,x,W_t,u_t,z_t)\mathrm{d}t-z_t\mathrm{d}W_t & \text{in } [0,T)\times\mathcal{D}, \\ \nabla\cdot u_t=0 & \text{in } [0,T)\times\mathcal{D} \end{cases} \tag{1.1}$$

with the associated terminal and boundary conditions

$$\begin{cases} u_T=\varphi(x,W_T) & \text{in } \mathcal{D}\times\mathbb{R}^q, \\ u_t=0 & \text{on } [0,T)\times\partial\mathcal{D}. \end{cases} \tag{1.2}$$

*Corresponding author. *Email addresses:* `wdzhao@sdu.edu.cn` (W. Zhao), `zhaowj@sdu.edu.cn` (W. Zhao), `syw1997@mail.sdu.edu.cn` (Y. Sun)

Here $\mathbb{F}=\{\mathcal{F}_t\}_{0\leq t\leq T}$ (T the final time instant) is the natural filtration of standard Brownian motion $W_t:=(W_t^1,W_t^2,\ldots,W_t^q)^\top$ and $\mathcal{F}_0$ containing all the $\mathbb{P}$-null sets of $\mathcal{F}$. $\mathcal{D}\subset\mathbb{R}^d$, $d=2,3$, is a bounded domain with Lipschitz continuous boundary, $\nu>0$ denotes the fluid viscosity. The velocity u, pressure p and auxiliary stochastic process z are unknown $\mathbb{F}$-adapted stochastic processes, i.e. $(u,p,z):[0,T]\times\mathcal{D}\times\Omega\rightarrow\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{d\times q}$, the function $f:[0,T]\times\mathcal{D}\times\mathbb{R}^q\times\mathbb{R}^d\times\mathbb{R}^{d\times q}\rightarrow\mathbb{R}^d$ and terminal condition $\varphi:\mathcal{D}\times\mathbb{R}^q\rightarrow\mathbb{R}^d$ are given functions.

The theory of the backward stochastic differential equations (BSDEs) is well studied in the past several decades and their related applications have been successfully made in the fields of stochastic optimal control, mathematical finance and nonlinear partial differential equations, etc., see [25, 30, 36, 39, 40, 42, 43] for details. By incorporating the physical principles into BSDEs, we come to the backward stochastic partial differential equations (BSPDEs), which also have a long research history, see [29]. As a specific case, BSSEs can be viewed as a stochastic backward evolution problem where the velocity profile at a final time instant T is observed and given. The importance of BSSEs can also be reflected by the stochastic optimal control problem in fluid dynamics. The existence and uniqueness of BSSEs have been studied in a nonlinear Navier-Stokes form by [32,34]. For general BSPDEs, many theoretical works are also conducted, see [2,9–12,18,20,21,24, 28,33,38] and references therein for details.

In the stochastic context, the stochastic Stokes equations usually consist of two types of models, that is, the forward stochastic Stokes equations (FSSEs) and BSSEs which are totally different since they inherit different stochastic natures [8, 30, 39]. The FSSEs with primary variables (u_t,p_t) are formulated in the forward direction of time. However, the BSSEs are formulated in the backward direction of time with a triple of unknown stochastic processes, i.e. (u_t,p_t,z_t) in (1.1)-(1.2). As for application, the FSSEs is concerned with how to recognize an objectively existing stochastic process and the BSSEs is mainly concerned with how to make a system achieve the desired goal in a randomly disturbed environment. In the view of control, the unknown process z_t plays a controlling role such that there exists a progressively measurable u_t satisfying the model (1.1) with a given random final condition u_T. Up to now, many significant contributions are already developed in the existing works for the FSSEs and their related models [3–7, 13, 14, 19, 22, 26, 27], etc. However, to our knowledge, there are no related numerical analysis works for BSSEs in the existing literature. Compared with FSSEs, BSSEs with an extra variable z have more complicated stochastic nature and larger computational complexity. Hence, to design the efficient numerical scheme and theoretical analysis methods for BSSEs, some new techniques should be considered. Therefore, BSSEs are worth receiving a separate study and analysis.

The finite element method as one of the popular numerical methodologies is successfully used for solving stochastic partial differential equations (SPDEs). In this paper, we focus our attentions on the finite element method for BSSEs. Beyond the theoretical techniques used for FSSEs, to address their own special stochastic properties of BSSEs, we seamlessly combine the techniques of finite element method, sub σ-algebra techniques and conditional expectation, etc, together. To solve the huge computational complexity