

Error Estimates of Finite Element Methods for the Nonlinear Backward Stochastic Stokes Equations

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Abstract. This paper is concerned with the numerical analyses of finite element methods for the nonlinear backward stochastic Stokes equations (BSSEs) where the forcing term is coupled with z . Under several developed analysis techniques, the error estimates of the proposed semi-discrete and fully discrete schemes, as well as their boundedness, are rigorously presented and established. Optimal convergence rates of the fully discrete scheme are obtained not only for the velocity u and auxiliary stochastic process z but also for the pressure p . For the efficiency of solving BSSEs, the proposed numerical scheme is parallelly designed in stochastic space. Numerical results are finally provided and tested in parallel to validate the theoretical results.

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1 Introduction

We consider a class of nonlinear backward stochastic Stokes equations defined on a complete and filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$

$$\begin{cases} -du_t - \nu \Delta u_t dt + \nabla p_t dt = f(t, x, W_t, u_t, z_t) dt - z_t dW_t & \text{in } [0, T) \times \mathcal{D}, \\ \nabla \cdot u_t = 0 & \text{in } [0, T) \times \mathcal{D} \end{cases} \quad (1.1)$$

with the associated terminal and boundary conditions

$$\begin{cases} u_T = \varphi(x, W_T) & \text{in } \mathcal{D} \times \mathbb{R}^q, \\ u_t = 0 & \text{on } [0, T) \times \partial \mathcal{D}. \end{cases} \quad (1.2)$$

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Here $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ (T the final time instant) is the natural filtration of standard Brownian motion $W_t := (W_t^1, W_t^2, \dots, W_t^q)^\top$ and $\mathcal{F}_0$ containing all the $\mathbb{P}$ -null sets of $\mathcal{F}$. $\mathcal{D} \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with Lipschitz continuous boundary, $\nu > 0$ denotes the fluid viscosity. The velocity u , pressure p and auxiliary stochastic process z are unknown $\mathbb{F}$ -adapted stochastic processes, i.e. $(u, p, z) : [0, T] \times \mathcal{D} \times \Omega \rightarrow \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{d \times q}$, the function $f : [0, T] \times \mathcal{D} \times \mathbb{R}^q \times \mathbb{R}^d \times \mathbb{R}^{d \times q} \rightarrow \mathbb{R}^d$ and terminal condition $\varphi : \mathcal{D} \times \mathbb{R}^q \rightarrow \mathbb{R}^d$ are given functions.

The theory of the backward stochastic differential equations (BSDEs) is well studied in the past several decades and their related applications have been successfully made in the fields of stochastic optimal control, mathematical finance and nonlinear partial differential equations, etc., see [25, 30, 36, 39, 40, 42, 43] for details. By incorporating the physical principles into BSDEs, we come to the backward stochastic partial differential equations (BSPDEs), which also have a long research history, see [29]. As a specific case, BSSEs can be viewed as a stochastic backward evolution problem where the velocity profile at a final time instant T is observed and given. The importance of BSSEs can also be reflected by the stochastic optimal control problem in fluid dynamics. The existence and uniqueness of BSSEs have been studied in a nonlinear Navier-Stokes form by [32, 34]. For general BSPDEs, many theoretical works are also conducted, see [2, 9–12, 18, 20, 21, 24, 28, 33, 38] and references therein for details.

In the stochastic context, the stochastic Stokes equations usually consist of two types of models, that is, the forward stochastic Stokes equations (FSSEs) and BSSEs which are totally different since they inherit different stochastic natures [8, 30, 39]. The FSSEs with primary variables (u_t, p_t) are formulated in the forward direction of time. However, the BSSEs are formulated in the backward direction of time with a triple of unknown stochastic processes, i.e. (u_t, p_t, z_t) in (1.1)–(1.2). As for application, the FSSEs is concerned with how to recognize an objectively existing stochastic process and the BSSEs is mainly concerned with how to make a system achieve the desired goal in a randomly disturbed environment. In the view of control, the unknown process z_t plays a controlling role such that there exists a progressively measurable u_t satisfying the model (1.1) with a given random final condition u_T . Up to now, many significant contributions are already developed in the existing works for the FSSEs and their related models [3–7, 13, 14, 19, 22, 26, 27], etc. However, to our knowledge, there are no related numerical analysis works for BSSEs in the existing literature. Compared with FSSEs, BSSEs with an extra variable z have more complicated stochastic nature and larger computational complexity. Hence, to design the efficient numerical scheme and theoretical analysis methods for BSSEs, some new techniques should be considered. Therefore, BSSEs are worth receiving a separate study and analysis.

The finite element method as one of the popular numerical methodologies is successfully used for solving stochastic partial differential equations (SPDEs). In this paper, we focus our attentions on the finite element method for BSSEs. Beyond the theoretical techniques used for FSSEs, to address their own special stochastic properties of BSSEs, we seamlessly combine the techniques of finite element method, sub σ -algebra techniques and conditional expectation, etc, together. To solve the huge computational complexity

of BSSEs, the proposed numerical schemes are efficiently designed to be synchronously parallel in the random space. The parallel techniques we proposed for BSSEs based on sub σ -algebras are totally different from those for FSSEs, where the numerical simulations are parallelly done pathwisely. For the coding, high performance parallel computing techniques with MPI-RMA techniques are employed. The main contributions of this paper are:

- (1) The proposed finite element methods for BSSEs are synchronously parallelizable in random space and can be efficiently solved.
- (2) The rigorously numerical analyses of BSSEs are provided. Optimal convergence rates of the fully discrete scheme are all obtained for (u, p, z) , respectively.
- (3) The numerical tests are realized on HPC cluster and demonstrate the feasibility of the proposed scheme.
- (4) This paper may bring some novel insights for solving more complicated backward stochastic models in fluid dynamics.

The reminder of this work is organized as follows. In Section 2, some useful spaces and notations are introduced, we define the variational solution of the BSSEs (1.1)-(1.2), several assumptions and lemmas are also given. In Section 3, we introduce finite element Galerkin approximation. Based on the variational solution, we present semi-discrete finite element scheme and give its error estimate in Section 4. Then we further propose a fully space-time discrete scheme for solving BSSEs (1.1)-(1.2) and give its numerical analysis results in Section 5. In Section 6, numerical tests are presented to verify our theoretical conclusions, followed by some conclusions in the final section.

2 Notation and preliminaries

For $m \in \mathbb{N}^+, 1 \leq p \leq \infty$, we use the notation $\mathcal{W}^{m,p}(\mathcal{D})$ for the Sobolev space with the norm $\|\cdot\|_{\mathcal{W}^{m,p}}$ (see [35]). Further, we denote Hilbert space $\mathcal{W}^{m,2}(\mathcal{D})$ by $H^m(\mathcal{D})$ with inner product $(\cdot, \cdot)_m$ and norm $\|\cdot\|_m$. Let $H_0^1(\mathcal{D})$ be the subspace of $H^1(\mathcal{D})$ with vanishing boundary condition on $\partial\mathcal{D}$, and $L^2(\mathcal{D})$ be the Lebesgue space of square integrable functions endowed with the inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$. In this paper, considering the inner products of vector-valued function space $(L^2(\mathcal{D}))^d$ (respectively $(H^m(\mathcal{D}))^d$) and matrix-valued function space $(L^2(\mathcal{D}))^{d \times q}$ (respectively $(H^m(\mathcal{D}))^{d \times q}$) are both defined as the sum of component-wise inner products, we let $(\cdot, \cdot)$ (respectively $(\cdot, \cdot)_m$) denote their corresponding inner products for notational simplicity, which can further induce the corresponding norms denoted by $\|\cdot\|$, respectively.

For the study of our model (1.1)-(1.2), the following spaces play fundamental roles:

$$X = (H_0^1(\mathcal{D}))^d, \quad Z = (H_0^1(\mathcal{D}))^{d \times q},$$

$$M = L_0^2(\mathcal{D}) = \left\{ q \in L^2(\mathcal{D}) : \int_{\mathcal{D}} q dx = 0 \right\},$$

$$V = \{ v \in X : (\nabla \cdot v, q) = 0, \forall q \in M \}.$$

In the stochastic setting, for a Hilbert space H , $L^2(\Omega, \mathcal{F}_t; H)$ denotes the space of $\mathcal{F}_t$ -measurable and H -valued stochastic variables ϕ satisfying $\mathbb{E}[\|\phi\|_H^2] < \infty$, where $\mathbb{E}[\cdot]$ is the expectation operator. Let $L_{\mathcal{F}}^p([0, T]; H)$ denote the space of all the $\mathbb{F}$ -adapted H -valued processes $\phi(t)$ satisfying

$$\mathbb{E} \left[\int_0^T \|\phi(t)\|_H^p dt \right] < \infty.$$

Moreover, we use $C_{\mathcal{F}}([0, T]; L^2(\Omega, H))$ to represent the space of all the $\mathbb{F}$ -adapted H -valued and mean square continuous processes satisfying $\sup_{0 \leq t \leq T} \mathbb{E}[\|\phi(t)\|_H^2] < \infty$.

In what follows, the assumptions of terminal condition φ and function f in (1.1) and (1.2) will be needed throughout the paper.

Assumption 2.1. Assume that the terminal condition φ is $\mathcal{F}_T$ -measurable and belongs to $L^2(\Omega, \mathcal{F}_T; (H^{k+2}(\mathcal{D}))^d)$, $k \geq 1$.

Assumption 2.2. The function f is uniformly Lipschitz continuous and satisfies the standard linear growth with positive constants L' and C'_l . For any $u, u' \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d \times q}$, we have

$$|f(t, x, \xi, u, z) - f(t, x, \xi, u', z')| \leq L'(|u - u'| + |z - z'|), \quad (2.1)$$

$$|f(t, x, \xi, u, z)| \leq C'_l(1 + |u| + |z|). \quad (2.2)$$

Let us introduce the Itô's formula for model (1.1)-(1.2).

Lemma 2.1 (Itô's Formula, [8, 39]). Assume that functional $\Phi: [0, T] \times L^2(\Omega, \mathcal{F}_t; H) \rightarrow \mathbb{R}$, and its partial derivatives $\Phi'_t, \Phi'_u, \Phi'_{uu}$ are uniformly continuous. Then for $t \in [0, T)$, there holds $\mathbb{P}$ -a.s.

$$\Phi(t, u_t) = \Phi(T, u_T) + \int_t^T \mathcal{L}[\Phi(s, u_s)] ds - \int_t^T \langle \Phi'_u(s, u_s), z_s dW_s \rangle \quad (2.3)$$

with

$$\begin{aligned} \mathcal{L}[\Phi(s, u_s)] &= -\Phi'_t(s, u_s) + \langle v \Delta u_s - \nabla p_s + f_s, \Phi'_u(s, u_s) \rangle \\ &\quad - \frac{1}{2} \langle \Phi''_{uu}(s, u_s) z_s, z_s \rangle. \end{aligned}$$

According to Itô's formula in Lemma 2.1, the boundedness of (u, z) is established in following lemma.

Lemma 2.2. Under Assumptions 2.1 and 2.2, suppose that $\mathbb{F}$ -adapted (u, p, z) valued in $X \times M \times (L^2(\mathcal{D}))^{d \times q}$ is a solution of BSSEs (1.1)-(1.2). Then there exists a positive constant $C > 0$ such that

$$\mathbb{E}[\|u_t\|^2] + 2v \int_t^T \mathbb{E}[\|\nabla u_s\|^2] ds + \frac{1}{2} \int_t^T \mathbb{E}[\|z_s\|^2] ds < C(\mathbb{E}[\|u_T\|^2] + T). \quad (2.4)$$

Proof. Applying Itô's formula in Lemma 2.1 to $\|u_t\|^2$, we have

$$\begin{aligned} \|u_t\|^2 + \int_t^T \|z_s\|^2 ds &= \|u_T\|^2 + \int_t^T (2u_s, \nu \Delta u_s - \nabla p_s + f_s) ds \\ &\quad - \int_t^T (2u_s, z_s dW_s). \end{aligned} \quad (2.5)$$

Using the integration by parts and taking mathematical expectation $\mathbb{E}[\cdot]$ in (2.5), we obtain

$$\begin{aligned} &\mathbb{E}[\|u_t\|^2] + 2\nu \int_t^T \mathbb{E}[\|\nabla u_s\|^2] ds + \int_t^T \mathbb{E}[\|z_s\|^2] ds \\ &= \mathbb{E}[\|u_T\|^2] + 2 \int_t^T \mathbb{E}[(\nabla \cdot u_s, p_s)] ds + 2 \int_t^T \mathbb{E}[(u_s, f_s)] ds. \end{aligned} \quad (2.6)$$

Under the linear growth of f in Assumption 2.2, it is clear that there is a constant $C_l > 0$ such that

$$\|f_s\|^2 \leq C_l(1 + \|u_s\|^2 + \|z_s\|^2). \quad (2.7)$$

With $\nabla \cdot u_s = 0$ and the Cauchy-Schwarz inequality, combining (2.6) and (2.7) gives

$$\begin{aligned} &\mathbb{E}[\|u_t\|^2] + 2\nu \int_t^T \mathbb{E}[\|\nabla u_s\|^2] ds + \int_t^T \mathbb{E}[\|z_s\|^2] ds \\ &= \mathbb{E}[\|u_T\|^2] + 2 \int_t^T \mathbb{E}[(u_s, f_s)] ds \\ &\leq \mathbb{E}[\|u_T\|^2] + \int_t^T \mathbb{E}[2C_l \|u_s\|^2] + \frac{1}{2C_l} \mathbb{E}[\|f_s\|^2] ds \\ &\leq \mathbb{E}[\|u_T\|^2] + \int_t^T \mathbb{E} \left[\left(2C_l + \frac{1}{2} \right) \|u_s\|^2 + \frac{1}{2} (\|z_s\|^2 + 1) \right] ds. \end{aligned} \quad (2.8)$$

It follows that

$$\begin{aligned} &\mathbb{E}[\|u_t\|^2] + 2\nu \int_t^T \mathbb{E}[\|\nabla u_s\|^2] ds + \frac{1}{2} \int_t^T \mathbb{E}[\|z_s\|^2] ds \\ &\leq \mathbb{E}[\|u_T\|^2] + \left(2C_l + \frac{1}{2} \right) \int_t^T \mathbb{E}[\|u_s\|^2] ds + \frac{T-t}{2}. \end{aligned} \quad (2.9)$$

Then, by backward Gronwall's inequality [39], the proof is completed. $\square$

We further state the Hölder continuity of u in time with the following lemmas.

Lemma 2.3. Under Assumptions 2.1 and 2.2, suppose that the $\mathbb{F}$ -adapted stochastic processes

$$\begin{aligned} (u, p, z) &\in C_{\mathcal{F}}([0, T]; L^2(\Omega, (H^{k+2}(\mathcal{D}))^d)) \times C_{\mathcal{F}}([0, T]; L^2(\Omega, (H^{k+1}(\mathcal{D}))^d)) \\ &\quad \times C_{\mathcal{F}}([0, T]; L^2(\Omega, (H^k(\mathcal{D}))^{d \times q})). \end{aligned}$$

Then for $t \in [s, T], s \geq 0$, there exists a positive constant C such that

$$\mathbb{E}[\|u_t - u_s\|_k^2] \leq C(t-s) \left(1 + \sup_{s \leq \tau \leq t} \mathbb{E}[\|u_\tau\|_{k+2}^2 + \|p_\tau\|_{k+1}^2 + \|z_\tau\|_k^2] \right). \quad (2.10)$$

Proof. Letting $\tau \in [s, t]$ and applying Itô's formula in [8] to $\|u_\tau - u_s\|_k^2$, one knows

$$\begin{aligned} \|u_t - u_s\|_k^2 &= -2 \int_s^t (u_\tau - u_s, \nu \Delta u_\tau - \nabla p_\tau + f_\tau)_k d\tau + \int_s^t \|z_\tau\|_k^2 d\tau \\ &\quad + 2 \int_s^t (u_\tau - u_s, z_\tau dW_\tau)_k. \end{aligned} \quad (2.11)$$

Taking the mathematical expectation $\mathbb{E}[\cdot]$ on (2.11) and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \mathbb{E}[\|u_t - u_s\|_k^2] &\leq (\nu + 2) \int_s^t \mathbb{E}[\|u_\tau - u_s\|_k^2] d\tau + \nu \int_s^t \mathbb{E}[\|\Delta u_\tau\|_k^2] d\tau \\ &\quad + \int_s^t \mathbb{E}[\|\nabla p_\tau\|_k^2] d\tau + \int_s^t \mathbb{E}[\|f_\tau\|_k^2] d\tau + \int_s^t \mathbb{E}[\|z_\tau\|_k^2] d\tau. \end{aligned} \quad (2.12)$$

Then (2.2) makes it obvious that

$$\begin{aligned} \mathbb{E}[\|u_t - u_s\|_k^2] &\leq C \left[\int_s^t \mathbb{E}[\|u_\tau - u_s\|_k^2] d\tau + \int_s^t \mathbb{E}[\|p_\tau\|_{k+1}^2] d\tau \right. \\ &\quad \left. + \int_s^t (1 + \mathbb{E}[\|u_\tau\|_{k+2}^2] + \mathbb{E}[\|z_\tau\|_k^2]) d\tau \right]. \end{aligned} \quad (2.13)$$

By the Gronwall's inequality, it follows that

$$\begin{aligned} \mathbb{E}[\|u_t - u_s\|_k^2] &\leq (1 + \mathbb{E}[\|u_\tau\|_{k+2}^2] + \mathbb{E}[\|p_\tau\|_{k+1}^2] + \mathbb{E}[\|z_\tau\|_k^2]) d\tau \\ &\leq C(t-s) \left(1 + \sup_{s \leq \tau \leq t} \mathbb{E}[\|u_\tau\|_{k+2}^2 + \|p_\tau\|_{k+1}^2 + \|z_\tau\|_k^2] \right), \end{aligned} \quad (2.14)$$

which ends proof. $\square$

For the design of the parallel algorithms for BSSEs, the following sub- σ algebra should be considered. We set $\mathcal{F}_t^{s, \xi}$ ($s \leq t \leq T$) as a σ -algebra generated by Brownian motion $\xi + \Delta W_{s, \tau}$ with $\Delta W_{s, \tau} = W_\tau - W_s, s \leq \tau \leq t$ starting from the time and stochastic space point (s, ξ) and set $\mathcal{F}^{s, \xi} := \mathcal{F}_T^{s, \xi}$. We denote $\mathbb{E}_t^{s, \xi}[\cdot]$ as the conditional expectation operator of the random variable under the σ -field $\mathcal{F}_t^{s, \xi}$, i.e. $\mathbb{E}_t^{s, \xi}[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t^{s, \xi}]$, and let $\mathbb{E}_t^\xi[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t^{t, \xi}]$. Further, according to Feynman-Kac formula in [31], there is a function $y(t, x, \xi) \in C^{1,2,2}$ such that $u(t, x) = y(t, x, W_t), z(t, x) = \nabla_\xi y(t, x, W_t)$, where $\nabla_\xi y$ denotes the gradient of y with respect to $\xi = (\xi_1, \xi_2, \dots, \xi_q)^\top$.

Lemma 2.4. For BSSEs (1.1)-(1.2), we assume that their solution

$$(u, p, z) \in L^2_{\mathcal{F}}([0, T]; (H^{k+4}(\mathcal{D}))^d) \times L^2_{\mathcal{F}}([0, T]; H^{k+3}(\mathcal{D})) \times L^2_{\mathcal{F}}([0, T]; (H^{k+2}(\mathcal{D}))^{d \times q}).$$

Under Assumptions 2.1 and 2.2, for any $s, t \in [0, T]$ with $s < t$, there exists a constant $C > 0$ such that

$$\|\mathbb{E}_s^{\tilde{\zeta}}[u_t - u_s]\|_{k+2}^2 \leq C(t-s) \int_s^t \mathbb{E}_s^{\tilde{\zeta}}[\|u_{\tau}\|_{k+4}^2 + \|p_{\tau}\|_{k+3}^2 + \|z_{\tau}\|_{k+2}^2] d\tau, \quad (2.15)$$

$$\|\mathbb{E}_s^{\tilde{\zeta}}[(u_t - u_s)\Delta W_{s,t}^{\top}]\|_{k+2}^2 \leq C(t-s) \int_s^t \mathbb{E}_s^{\tilde{\zeta}}[\|u_{\tau}\|_{k+4}^2 + \|p_{\tau}\|_{k+3}^2 + \|z_{\tau}\|_{k+2}^2] d\tau. \quad (2.16)$$

Proof. The proof can be directly completed with the similar techniques as those in Lemma 2.3. For further details, we refer to [33]. $\square$

To formulate the BSSEs (1.1)-(1.2) in the variational framework and provide the definition of a variational solution. We multiply (1.1) with test functions (v, q) in $X \times M$ and integrate over the space domain. Using the integration by parts with respect to the spatial variable, the variational solution of (1.1)-(1.2) is given as follows.

Definition 2.1 (Variational Solution). A triple of $\mathbb{F}$ -adapted stochastic processes

$$(u, p, z) \in (C_{\mathcal{F}}([0, T]; L^2(\Omega, X)) \cap L^2_{\mathcal{F}}([0, T]; X)) \times L^2_{\mathcal{F}}([0, T]; M) \times L^2_{\mathcal{F}}([0, T]; (L^2(\mathcal{D}))^{d \times q})$$

is called a variational solution of BSSEs (1.1)-(1.2) if it satisfies almost surely for all $t \in [0, T]$, $v \in X$ and $q \in M$

$$\begin{aligned} & (u_t, v) + \int_t^T v(\nabla u_s, \nabla v) ds - \int_t^T (p_s, \nabla \cdot v) ds \\ &= (u_T, v) + \int_t^T (f_s, v) ds - \left(\int_t^T z_s dW_s, v \right), \end{aligned} \quad (2.17)$$

$$(\nabla \cdot u_t, q) = 0. \quad (2.18)$$

In addition to lemmas above, we also provide a useful lemma that will be needed subsequently.

Lemma 2.5 (Discrete Backward Gronwall's Inequality, [39, 44]). Let $x_m, b_m, c_m \geq 0$ and $a_m > 0$ for $m = N, N-1, \dots, 1$, and let the sequence a_m be nondecreasing. Then if

$$x_m + c_m \leq a_m + \sum_{n=m+1}^N \Delta t b_n x_n,$$

we have

$$x_m + c_m \leq a_m \exp \left(\Delta t \prod_{n=m}^{N-1} b_{n+1} \right). \quad (2.19)$$

3 Finite element Galerkin approximation

Let $\{\mathcal{T}_h\}$ be a family of regular triangulations of $\mathcal{D}$ with maximum mesh size $h > 0$ and $\mathcal{P}_k(K)$ denote the set of polynomials of degree less than or equal to $k \in \mathbb{N}^+$ on element $K \in \mathcal{T}_h$. We define the finite element spaces $(X_h, M_h, V_h, \mathbb{Z}_h) \subset (X, M, V, \mathbb{Z})$ as

$$\begin{aligned} X_h &= \{v_h \in (C^0(\mathcal{D}))^d \cap X : v_h|_K \in (\mathcal{P}_{k+1}(K))^d, \forall K \in \mathcal{T}_h\}, \\ M_h &= \{q_h \in C^0(\mathcal{D}) \cap M : q_h|_K \in \mathcal{P}_k(K), \forall K \in \mathcal{T}_h\}, \\ V_h &= \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0, \forall q_h \in M_h\}, \\ \mathbb{Z}_h &= \{z_h \in (C^0(\mathcal{D}))^{d \times q} \cap \mathbb{Z} : z_h|_K \in (\mathcal{P}_{k+1}(K))^{d \times q}, \forall K \in \mathcal{T}_h\} \end{aligned}$$

such that the discrete Ladyzhenskaya-Babuška-Brezzi (LBB) condition [35] is satisfied

$$\sup_{\substack{v_h \in X_h \\ v_h \neq 0}} \frac{(\nabla \cdot v_h, q_h)}{\|\nabla v_h\|} \geq \bar{C} \|q_h\|, \quad \forall q_h \in M_h, \quad (3.1)$$

where $\bar{C}$ is a positive constant.

Referring to [16, 17, 35], we introduce Stokes projection $\mathcal{S}_h : X \times M \rightarrow X_h \times M_h$ such that

$$\begin{cases} \nu(\nabla(u - \mathcal{S}_h u), \nabla v_h) - (p - \mathcal{S}_h p, \nabla \cdot v_h) = 0, & \forall v_h \in X_h, \\ (\nabla \cdot \mathcal{S}_h u, q_h) = 0, & \forall q_h \in M_h, \end{cases} \quad (3.2)$$

and L^2 -orthogonal projections $\mathcal{P}_h : (L^2(\mathcal{D}))^d \rightarrow V_h$ and $\mathbb{P}_h : (L^2(\mathcal{D}))^{d \times q} \rightarrow (V_h)^q$ as

$$(u, v_h) = (\mathcal{P}_h u, v_h), \quad \forall v_h \in V_h, \quad (3.3)$$

$$(z, w_h) = (\mathbb{P}_h z, w_h), \quad \forall w_h \in (V_h)^q. \quad (3.4)$$

The operators $\mathcal{S}_h, \mathcal{P}_h$ and $\mathbb{P}_h$ follow the classical approximation properties (see [16, 23]). For all $u \in (H^{k+2}(\mathcal{D}))^d, p \in H^{k+1}(\mathcal{D}), z \in (H^{k+2}(\mathcal{D}))^{d \times q}$, there is positive constant C independent of h such that

$$\|u - \mathcal{S}_h u\| + h(\|\nabla(u - \mathcal{S}_h u)\| + \|p - \mathcal{S}_h p\|) \leq Ch^{k+2}(\|u\|_{k+2} + \|p\|_{k+1}), \quad (3.5)$$

$$\|u - \mathcal{P}_h u\| + h\|\nabla(u - \mathcal{P}_h u)\| \leq Ch^{k+2}\|u\|_{k+2}, \quad (3.6)$$

$$\|z - \mathbb{P}_h z\| + h\|\nabla(z - \mathbb{P}_h z)\| \leq Ch^{k+2}\|z\|_{k+2}. \quad (3.7)$$

4 Semi-discrete finite element method

Based on the variational form (2.17), a semi-discrete finite element scheme in space for solving BSSEs (1.1)-(1.2) is given as follows.

Scheme 4.1. Given terminal conditions

$$u_h(T) = \mathcal{S}_h u(T), \quad p_h(T) = \mathcal{S}_h p(T), \quad z_h(T) = \mathbb{P}_h z(T)$$

solve for

$$(u_h, p_h, z_h) \in (C_{\mathcal{F}}([0, T]; L^2(\Omega, X_h)) \cap L^2_{\mathcal{F}}([0, T]; X_h)) \times L^2_{\mathcal{F}}([0, T]; M_h) \times L^2_{\mathcal{F}}([0, T]; \mathbb{Z}_h)$$

with boundary conditions $u_h(t) = 0, z_h(t) = 0$ on $\partial\mathcal{D}$ such that for all $t \in [0, T], v_h \in X_h, q_h \in M_h$,

$$\begin{aligned} & (u_h(t), v_h) + \int_t^T v(\nabla u_h(s), \nabla v_h) ds - \int_t^T (p_h(s), \nabla \cdot v_h) ds \\ &= (u_h(T), v_h) + \int_t^T (f_h(s), v_h) ds - \left(\int_t^T z_h(s) dW_s, v_h \right), \end{aligned} \quad (4.1)$$

$$(\nabla \cdot u_h(t), q_h) = 0, \quad (4.2)$$

where we denote $f(s, x, W_s, u_h(s), z_h(s))$ by $f_h(s)$ for simplicity.

To provide the boundedness of Scheme 4.1, we need to introduce the following discrete operator $A_h: V_h \rightarrow V_h$ defined by

$$(A_h v_h, w_h) = -(\nabla v_h, \nabla w_h), \quad \forall v_h, w_h \in V_h. \quad (4.3)$$

Then the semi-discrete problem (4.1)-(4.2) is equivalent to find $u_h \in L^2_{\mathcal{F}}([0, T]; V_h), z_h \in L^2_{\mathcal{F}}([0, T]; \mathbb{Z}_h)$ such that

$$\begin{cases} u_h(t) = u_h(T) + \int_t^T v A_h u_h ds + \int_t^T \mathcal{P}_h f_h(s) ds - \int_t^T \mathbb{P}_h z_h dW_s, \\ u_h(T) = \mathcal{S}_h u(T), \quad z_h(T) = \mathbb{P}_h z(T). \end{cases} \quad (4.4)$$

Theorem 4.1 (Boundedness). *Suppose Assumptions 2.1 and 2.2 hold and $(u_h(t), p_h(t), z_h(t))$ is a solution of semi-discrete Scheme 4.1. Then there is a positive constant C such that*

$$\mathbb{E}[\|u_h(t)\|^2] + \int_t^T \left(v \mathbb{E}[\|\nabla u_h(s)\|^2] + \mathbb{E}[\|z_h(s)\|^2] \right) ds \leq C(\|u_h(T)\|^2 + T). \quad (4.5)$$

Proof. Considering (4.4) and applying Itô's formula to $\|u_h(t)\|^2$, with the backward Gronwall's inequality [33], we can directly end the proof. $\square$

4.1 Convergence in space for Scheme 4.1

To derive the error estimates of semi-discrete finite Scheme 4.1, we introduce the following notations:

$$e_u^t = u_t - u_h(t), \quad e_z^t = z_t - z_h(t),$$

$$\begin{aligned}\rho_u^t &= u_t - \mathcal{S}_h u_t, & \theta_u^t &= \mathcal{S}_h u_t - u_h(t), \\ \rho_z^t &= z_t - \mathbb{P}_h z_t, & \theta_z^t &= \mathbb{P}_h z_t - z_h(t), \\ e_f^t &= f_t - f_h(t) = f(t, x, W_t, u_t, z_t) - f(t, x, W_t, u_h(t), z_h(t)).\end{aligned}$$

Now, we are ready to give the error estimates of Scheme 4.1 in the following theorem.

Theorem 4.2. *Assume that Assumptions 2.1 and 2.2 hold and the variational solution of (2.17) satisfies*

$$\begin{aligned}(u, p, z) &\in \left(C_{\mathcal{F}} \left([0, T]; L^2(\Omega, X \cap (H^{k+4}(\mathcal{D}))^d) \right) \cap L_{\mathcal{F}}^2([0, T]; X \cap (H^{k+4}(\mathcal{D}))^d) \right) \\ &\quad \times L_{\mathcal{F}}^2([0, T]; M \cap H^{k+3}(\mathcal{D})) \times L_{\mathcal{F}}^2([0, T]; \mathbb{Z} \cap (H^{k+2}(\mathcal{D}))^{d \times q}).\end{aligned}$$

Then for $\mathbb{F}$ -adapted random field (u_h, p_h, z_h) , by setting $u_h(T) = \mathcal{S}_h u(T)$, and $z_h(T) = \mathbb{P}_h z(T)$ in Scheme 4.1, there exists a positive constant C that is independent of h such that

$$\mathbb{E}[\|e_u^t\|^2] + \nu \int_t^T \mathbb{E}[\|\nabla e_u^s\|^2] ds + \frac{1}{2} \int_t^T \mathbb{E}[\|e_z^s\|^2] ds \leq Ch^{2(k+1)}. \quad (4.6)$$

Proof. Setting $v = v_h$ in (2.17) and (4.1), separately, and subtracting (4.1) from (2.17) yield

$$\begin{aligned}& (e_u^t, v_h) + \int_t^T \nu (\nabla \theta_u^s, \nabla v_h) ds - \int_t^T (\mathcal{S}_h p_s - p_h(s), \nabla \cdot v_h) ds \\ &= (e_u^T, v_h) + \int_t^T (e_f^s, v_h) ds - \left(\int_t^T e_z^s dW_s, v_h \right).\end{aligned} \quad (4.7)$$

According to the definitions of the discrete operator A_h , L^2 -orthogonal operators $\mathcal{P}_h$ and $\mathbb{P}_h$, (4.7) can be equivalently written into the following form:

$$\mathcal{P}_h e_u^t + \int_t^T -\nu A_h \theta_u^s ds = \mathcal{P}_h e_u^T + \int_t^T \mathcal{P}_h e_f^s ds - \int_t^T \mathbb{P}_h e_z^s dW_s. \quad (4.8)$$

By applying Itô's formula to $\|\mathcal{P}_h e_u^t\|^2$, it follows that

$$\begin{aligned}& \|\mathcal{P}_h e_u^t\|^2 + \int_t^T \|\mathbb{P}_h e_z^s\|^2 ds + 2\nu \int_t^T (\nabla \mathcal{P}_h e_u^s, \nabla \theta_u^s) ds \\ &= \|\mathcal{P}_h e_u^T\|^2 + 2 \int_t^T (\mathcal{P}_h e_u^s, \mathcal{P}_h e_f^s) ds - 2 \int_t^T (\mathcal{P}_h e_u^s, \mathbb{P}_h e_z^s dW_s).\end{aligned} \quad (4.9)$$

Here, we note that $\mathcal{P}_h e_u^s = \mathcal{P}_h u_s - u_h(s)$, $\mathbb{P}_h e_z^s = \mathbb{P}_h z_s - z_h(s)$, and define

$$\begin{aligned}\tilde{\rho}_u^s &:= u_s - \mathcal{P}_h u_s, & \tilde{\theta}_u^s &:= \mathcal{P}_h u_s - u_h(s), \\ \tilde{\rho}_z^s &:= z_s - \mathbb{P}_h z_s, & \tilde{\theta}_z^s &:= \mathbb{P}_h z_s - z_h(s).\end{aligned}$$

One can check that the following equality holds under the definition of e_u^s :

$$(\nabla \mathcal{P}_h e_u^s, \nabla \theta_u^s) = (\nabla \tilde{\theta}_u^s, \nabla \theta_u^s) = -(\nabla \tilde{\rho}_u^s, \nabla \theta_u^s) + (\nabla \theta_u^s, \nabla \theta_u^s) + (\nabla \rho_u^s, \nabla \theta_u^s). \quad (4.10)$$

By above equality and taking $\mathbb{E}[\cdot]$ to (4.9), then we get

$$\begin{aligned} & \mathbb{E}[\|\tilde{\theta}_u^t\|^2] + \int_t^T \mathbb{E}[\|\tilde{\theta}_z^s\|^2] ds + 2\nu \int_t^T \mathbb{E}[\|\nabla \theta_u^s\|^2] ds \\ &= \mathbb{E}[\|\tilde{\theta}_u^T\|^2] + 2\nu \int_t^T (\mathbb{E}[(\nabla \tilde{\rho}_u^s, \nabla \theta_u^s)] - \mathbb{E}[(\nabla \rho_u^s, \nabla \theta_u^s)]) ds \\ & \quad + 2 \int_t^T \mathbb{E}[(\tilde{\theta}_u^s, \mathcal{P}_h e_f^s)] ds. \end{aligned} \quad (4.11)$$

It remains to estimate the terms on right side of (4.11). By applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} 2\nu \int_t^T \mathbb{E}[(\nabla \tilde{\rho}_u^s, \nabla \theta_u^s)] ds &\leq 2\nu \int_t^T \mathbb{E}[\|\nabla \tilde{\rho}_u^s\| \cdot \|\nabla \theta_u^s\|] ds \\ &\leq \nu \int_t^T \mathbb{E} \left[2\|\nabla \tilde{\rho}_u^s\|^2 + \frac{\|\nabla \theta_u^s\|^2}{2} \right] ds, \end{aligned} \quad (4.12)$$

$$\begin{aligned} 2\nu \int_t^T \mathbb{E}[(\nabla \rho_u^s, \nabla \theta_u^s)] ds &\leq 2\nu \int_t^T \mathbb{E}[\|\nabla \rho_u^s\| \cdot \|\nabla \theta_u^s\|] ds \\ &\leq \nu \int_t^T \mathbb{E} \left[2\|\nabla \rho_u^s\|^2 + \frac{\|\nabla \theta_u^s\|^2}{2} \right] ds. \end{aligned} \quad (4.13)$$

Due to the Lipschitz condition of f , a direct computation shows that there is a positive constant L such that

$$\|e_f^t\| \leq L(\|e_u^t\| + \|e_z^t\|). \quad (4.14)$$

Since $\|\mathcal{P}_h e_f^s\| \leq \|e_f^s\|$, then we get

$$\begin{aligned} 2\mathbb{E}[(\tilde{\theta}_u^s, \mathcal{P}_h e_f^s)] &\leq 2L\mathbb{E}[\|\tilde{\theta}_u^s\|(\|\tilde{\rho}_u^s\| + \|\tilde{\theta}_u^s\| + \|\tilde{\rho}_z^s\| + \|\tilde{\theta}_z^s\|)] \\ &\leq 2L(2L+1)\mathbb{E}[\|\tilde{\theta}_u^s\|^2] + \mathbb{E} \left[\|\tilde{\rho}_u^s\|^2 + \|\tilde{\rho}_z^s\|^2 + \frac{\|\tilde{\theta}_z^s\|^2}{2} \right]. \end{aligned} \quad (4.15)$$

Substituting (4.12), (4.13) and (4.15) into (4.11), it follows that

$$\begin{aligned} & \mathbb{E}[\|\tilde{\theta}_u^t\|^2] + \frac{1}{2} \int_t^T \mathbb{E}[\|\tilde{\theta}_z^s\|^2] ds + \nu \int_t^T \mathbb{E}[\|\nabla \theta_u^s\|^2] ds \\ &\leq \mathbb{E}[\|\tilde{\theta}_u^T\|^2] + 2L(2L+1) \int_t^T \mathbb{E}[\|\tilde{\theta}_u^s\|^2] ds + 2\nu \int_t^T \mathbb{E}[\|\nabla \tilde{\rho}_u^s\|^2] ds \\ & \quad + 2\nu \int_t^T \mathbb{E}[\|\nabla \rho_u^s\|^2] ds + \int_t^T \mathbb{E}[\|\tilde{\rho}_u^s\|^2] ds + \int_t^T \mathbb{E}[\|\tilde{\rho}_z^s\|^2] ds. \end{aligned} \quad (4.16)$$

Then by the backward Gronwall's inequality one gets

$$\begin{aligned} & \mathbb{E}[\|\tilde{\theta}_u^t\|^2] + \frac{1}{2} \int_t^T \mathbb{E}[\|\tilde{\theta}_z^s\|^2] ds + \nu \int_t^T \mathbb{E}[\|\nabla \theta_u^s\|^2] ds \\ & \leq C \left(\mathbb{E}[\|\tilde{\theta}_u^T\|^2] + \nu \int_t^T \mathbb{E}[\|\nabla \tilde{\rho}_u^s\|^2] ds + \nu \int_t^T \mathbb{E}[\|\nabla \rho_u^s\|^2] ds \right. \\ & \quad \left. + \int_t^T \mathbb{E}[\|\tilde{\rho}_u^s\|^2] ds + \int_t^T \mathbb{E}[\|\tilde{\rho}_z^s\|^2] ds \right). \end{aligned} \quad (4.17)$$

Consequently, using triangle inequality we can derive that

$$\begin{aligned} & \mathbb{E}[\|e_u^t\|^2] + \frac{1}{2} \int_t^T \mathbb{E}[\|e_z^s\|^2] ds + \nu \int_t^T \mathbb{E}[\|\nabla e_u^s\|^2] ds \\ & \leq C \left(\mathbb{E}[\|\tilde{\theta}_u^T\|^2] + \mathbb{E}[\|\tilde{\rho}_u^T\|^2] + \int_t^T \mathbb{E}[\|\tilde{\rho}_u^s\|^2] ds + \int_t^T \mathbb{E}[\|\tilde{\rho}_z^s\|^2] ds \right. \\ & \quad \left. + \nu \int_t^T \mathbb{E}[\|\nabla \rho_u^s\|^2] ds + \nu \int_t^T \mathbb{E}[\|\nabla \tilde{\rho}_u^s\|^2] ds \right) \\ & \leq Ch^{2(k+1)}, \end{aligned} \quad (4.18)$$

which establishes the error estimate (4.6) by the properties of operators $\mathcal{S}_h, \mathcal{P}_h$ and $\mathbb{P}_h$ in (3.5) and (3.6). $\square$

5 Fully discrete finite element method

In this section, we shall further discrete the variational form (2.17) of BSSEs in time to propose a fully discrete finite element scheme, then we present its boundedness and error estimates. To this end, we first introduce a time partition. For the time interval $[0, T]$, $N \in \mathbb{N}^+$, we consider a time partition $\pi_t : 0 = t_0 < t_1 < \dots < t_N = T$ with a regularity constraint

$$\frac{\max_{0 \leq n \leq N-1} \Delta t_n}{\min_{0 \leq n \leq N-1} \Delta t_n} \leq C_0, \quad (5.1)$$

where $C_0 > 0, \Delta t_n := t_{n+1} - t_n$ with $\Delta t = \max_{0 \leq n \leq N-1} \Delta t_n$. For a standard Brownian motion W_t , we denote the increment $W_{t_{n+1}} - W_{t_n}$ by $\Delta W_{t_{n+1}}$ in following sequel.

The fully discrete numerical scheme will be given by the following procedure on the regular time partition π_t . We consider the integral form (2.17) of BSSEs from time instant t_n to t_{n+1} for all $v \in X, q \in M$,

$$\begin{aligned} & (u_{t_n}, v) + \int_{t_n}^{t_{n+1}} \nu (\nabla u_t, \nabla v) - (p_t, \nabla \cdot v) dt \\ & = (u_{t_{n+1}}, v) + \int_{t_n}^{t_{n+1}} (f_t, v) dt - \left(\int_{t_n}^{t_{n+1}} z_t dW_t, v \right), \end{aligned} \quad (5.2)$$

$$(\nabla \cdot u_{t_n}, q) = 0. \quad (5.3)$$

Taking the conditional expectations $\mathbb{E}_{t_n}^{\tilde{\cdot}}[\cdot]$ of (5.2) and (5.3), we have

$$\begin{aligned} & (u_{t_n}, v) + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\tilde{\cdot}} [\nu(\nabla u_t, \nabla v) - (p_t, \nabla \cdot v)] dt \\ &= \left(\mathbb{E}_{t_n}^{\tilde{\cdot}} [u_{t_{n+1}}], v \right) + \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^{\tilde{\cdot}} [f_t], v \right) dt, \end{aligned} \quad (5.4)$$

$$(\nabla \cdot u_{t_n}, q) = 0. \quad (5.5)$$

Estimating the definite integrals in (5.4) with left rectangle rule, then we obtain

$$\begin{aligned} & (u_{t_n}, v) + \nu \Delta t_n (\nabla u_{t_n}, \nabla v) - \Delta t_n (p_{t_n}, \nabla \cdot v) \\ &= \left(\mathbb{E}_{t_n}^{\tilde{\cdot}} [u_{t_{n+1}}], v \right) + \Delta t_n (f_{t_n}, v) + (R_u^n, v), \end{aligned} \quad (5.6)$$

where $R_u^n = R_1^n + R_2^n + R_3^n$ is a truncation error with

$$R_1^n = \int_{t_n}^{t_{n+1}} \nu \mathbb{E}_{t_n}^{\tilde{\cdot}} [\Delta u_t] dt - \nu \Delta t_n \Delta u_{t_n}, \quad (5.7)$$

$$R_2^n = - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\tilde{\cdot}} [\nabla p_t] dt + \Delta t_n \nabla p_{t_n}, \quad (5.8)$$

$$R_3^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\tilde{\cdot}} [f_t] dt - \Delta t_n f_{t_n}. \quad (5.9)$$

We multiply the integral form of (1.1) from t_n to t_{n+1} by $\Delta W_{t_{n+1}}$, and further multiply derived equation with test functions $w \in \mathbb{Z}$. Then integrating the corresponding equations over spatial domain $\mathcal{D}$, we have

$$\begin{aligned} & (u_{t_n} \Delta W_{t_{n+1}}^\top, w) + \nu \left(\int_{t_n}^{t_{n+1}} \nabla u_t dt \Delta W_{t_{n+1}}^\top, \nabla w \right) - \left(\int_{t_n}^{t_{n+1}} p_t dt \Delta W_{t_{n+1}}^\top, \nabla \cdot w \right) \\ &= (u_{t_{n+1}} \Delta W_{t_{n+1}}^\top, w) + \left(\int_{t_n}^{t_{n+1}} f_t dt \Delta W_{t_{n+1}}^\top, w \right) - \left(\int_{t_n}^{t_{n+1}} z_t dW_t \Delta W_{t_{n+1}}^\top, w \right). \end{aligned} \quad (5.10)$$

Taking the conditional expectations $\mathbb{E}_{t_n}^{\tilde{\cdot}}[\cdot]$ on both sides of (5.10), with the Martingale property, Itô isometry and left rectangle rule for approximating the integrals in (5.10), we then obtain

$$0 = \left(\mathbb{E}_{t_n}^{\tilde{\cdot}} [u_{t_{n+1}} \Delta W_{t_{n+1}}^\top], w \right) - \Delta t_n (z_{t_n}, w) + (R_z^n, w), \quad (5.11)$$

where

$$R_z^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{\tilde{\cdot}} [(f_t + \nu \Delta u_t - \nabla p_t) \Delta W_{t_{n+1}}^\top] dt - \int_{t_n}^{t_{n+1}} \left(\mathbb{E}_{t_n}^{\tilde{\cdot}} [z_t] - z_{t_n} \right) dt. \quad (5.12)$$

Now we are in a position to propose our fully discrete numerical scheme of (1.1)-(1.2) in physical space and time. We denote $f(t_n, x, W_{t_n}, u_h^n, z_h^n)$ by f_h^n . Based on (5.6), (5.5) and (5.11), omitting the truncation error terms, we propose the following scheme for solving BSSEs (2.17).

Scheme 5.1. Given terminal condition $u_h^N = \mathcal{S}_h u(T)$, solve for a triple of $\mathbb{F}$ -adapted $X_h \times M_h \times \mathbb{Z}_h$ -valued random processes $(u_h^n, p_h^n, z_h^n), n = N-1, \dots, 0$ with boundary conditions $u_h^N = 0, z_h^N = 0$ on $\partial\mathcal{D}$ such that for all $v_h \in X_h, q_h \in M_h$ and $w_h \in \mathbb{Z}_h$,

$$(u_h^n, v_h) + \Delta t_n \nu (\nabla u_h^n, \nabla v_h) - \Delta t_n (p_h^n, \nabla \cdot v_h) = \left(\mathbb{E}_{t_n}^\xi [u_h^{n+1}] + \Delta t_n f_h^n, v_h \right), \quad (5.13)$$

$$(\nabla \cdot u_h^n, q_h) = 0, \quad (5.14)$$

$$\Delta t_n (z_h^n, w_h) = \left(\mathbb{E}_{t_n}^\xi [u_h^{n+1} \Delta W_{t_{n+1}}^\top], w_h \right). \quad (5.15)$$

Remark 5.1. Given time instant t_n , Scheme 5.1 can be parallelly performed in the random space for $\xi \in \mathbb{R}^q$ with the MPI-RMA techniques.

To see the stability property of Scheme 5.1, we set $\mathcal{F}_{t_n}$ -adapted $\varepsilon_f^{n,h}$ and $\mathcal{F}_T$ -measurable $\varepsilon_u^{N,h}$ as perturbations on the f_h^n and terminal condition u_h^N , respectively. Let

$$f_h^{n,\varepsilon} = f(t_n, x, W_{t_n}, u_h^{n,\varepsilon}, z_h^{n,\varepsilon}).$$

The perturbed numerical solution $(u_h^{n,\varepsilon}, p_h^{n,\varepsilon}, z_h^{n,\varepsilon})$ under the perturbed conditions satisfies

$$\begin{aligned} & (u_h^{n,\varepsilon}, v_h) + \Delta t_n \nu (\nabla u_h^{n,\varepsilon}, \nabla v_h) - \Delta t_n (p_h^{n,\varepsilon}, \nabla \cdot v_h) \\ &= \left(\mathbb{E}_{t_n}^\xi [u_h^{n+1,\varepsilon}], v_h \right) + \Delta t_n (f_h^{n,\varepsilon}, v_h) + \Delta t_n (\varepsilon_f^{n,h}, v_h), \end{aligned} \quad (5.16)$$

$$(\nabla \cdot u_h^{n,\varepsilon}, q_h) = 0, \quad (5.17)$$

$$\Delta t_n (z_h^{n,\varepsilon}, w_h) = \left(\mathbb{E}_{t_n}^\xi [u_h^{n+1,\varepsilon} \Delta W_{t_{n+1}}^\top], w_h \right). \quad (5.18)$$

Substituting (5.13)-(5.15) from (5.16)-(5.18), respectively, we obtain

$$\begin{aligned} & (\varepsilon_u^{n,h}, v_h) + \Delta t_n \nu (\nabla \varepsilon_u^{n,h}, \nabla v_h) - \Delta t_n (\varepsilon_p^{n,h}, \nabla \cdot v_h) \\ &= \left(\mathbb{E}_{t_n}^\xi [\varepsilon_u^{n+1,h}], v_h \right) + \Delta t_n (f_h^{n,\varepsilon} - f_h^n, v_h) + \Delta t_n (\varepsilon_f^{n,h}, v_h), \end{aligned} \quad (5.19)$$

$$(\nabla \cdot \varepsilon_u^{n,h}, q_h) = 0, \quad (5.20)$$

$$\Delta t_n (\varepsilon_z^{n,h}, w_h) = \left(\mathbb{E}_{t_n}^\xi [\varepsilon_u^{n+1,h} \Delta W_{t_{n+1}}^\top], w_h \right) \quad (5.21)$$

with $\varepsilon_u^{n,h} = u_h^{n,\varepsilon} - u_h^n$, $\varepsilon_z^{n,h} = z_h^{n,\varepsilon} - z_h^n$, $\varepsilon_p^{n,h} = p_h^{n,\varepsilon} - p_h^n$.

We call (5.19)-(5.21) the permutation error equations for Scheme 5.1. Now we present the stability result as follows.

Theorem 5.1 (Stability). Under Assumptions 2.1 and 2.2, let (u_h^n, p_h^n, z_h^n) be the solution of Scheme 5.1. There exists a positive constant C such that for a sufficiently small $\Delta t > 0$,

$$\begin{aligned} & \mathbb{E} [\|\varepsilon_u^{N,h}\|^2] + \sum_{i=n}^{N-1} \nu \Delta t \mathbb{E} [\|\nabla \varepsilon_u^{i,h}\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|\varepsilon_z^{i,h}\|^2] \\ & \leq C \left(\mathbb{E} [\|\varepsilon_u^{N,h}\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|\varepsilon_f^{i,h}\|^2] \right). \end{aligned} \quad (5.22)$$

Proof. By setting $v_h = \varepsilon_u^{n,h}, q_h = \varepsilon_p^{n,h}$ in (5.19)-(5.20), respectively, considering $(\nabla \cdot \varepsilon_u^{n,h}, \varepsilon_p^{n,h}) = 0$ and using Cauchy-Schwarz inequality and Lipschitz condition of f in (2.1), we get

$$\begin{aligned} & \|\varepsilon_u^{n,h}\|^2 + \nu \Delta t_n \|\nabla \varepsilon_u^{n,h}\|^2 \\ &= \left(\mathbb{E}_{t_n}^{\zeta} [\varepsilon_u^{n+1,h}], \varepsilon_u^{n,h} \right) + \Delta t_n (f_h^{n,\varepsilon} - f_h^n, \varepsilon_u^{n,h}) + \Delta t_n (\varepsilon_f^{n,h}, \varepsilon_u^{n,h}) \\ &\leq \frac{\|\varepsilon_u^{n,h}\|^2}{2} + \frac{\|\mathbb{E}_{t_n}^{\zeta} [\varepsilon_u^{n+1,h}]\|^2}{2} + \frac{\Delta t_n}{4} (\|\varepsilon_u^{n,h}\|^2 + \|\varepsilon_z^{n,h}\|^2) \\ &\quad + \frac{\Delta t_n}{2} \|\varepsilon_u^{n,h}\|^2 + \frac{\Delta t_n}{2} \|\varepsilon_f^{n,h}\|^2 + L \Delta t_n \|\varepsilon_u^{n,h}\|^2. \end{aligned} \quad (5.23)$$

We further reorganize (5.23) to obtain

$$\begin{aligned} & (1 - (1+2L)\Delta t_n) \|\varepsilon_u^{n,h}\|^2 + 2\nu \Delta t_n \|\nabla \varepsilon_u^{n,h}\|^2 \\ &\leq \left\| \mathbb{E}_{t_n}^{\zeta} [\varepsilon_u^{n+1,h}] \right\|^2 + \frac{\Delta t_n}{2} \|\varepsilon_z^{n,h}\|^2 + \Delta t_n \|\varepsilon_f^{n,h}\|^2. \end{aligned} \quad (5.24)$$

Similarly, setting $w_h = \varepsilon_z^{n,h}$ in (5.21) and applying Cauchy-Schwarz inequality, it holds that

$$\Delta t_n \|\varepsilon_z^{n,h}\|^2 \leq \mathbb{E}_{t_n}^{\zeta} [\|\varepsilon_u^{n+1,h}\|^2] - \left\| \mathbb{E}_{t_n}^{\zeta} [\varepsilon_u^{n+1,h}] \right\|^2. \quad (5.25)$$

Here we use the property

$$\begin{aligned} & \left\| \mathbb{E}_{t_n}^{\zeta} [\varepsilon_u^{n+1,h} \Delta W_{t_{n+1}}^{\top}] \right\|^2 = \left\| \mathbb{E}_{t_n}^{\zeta} [(\varepsilon_u^{n+1,h} - \mathbb{E}_{t_n}^{\zeta} [\varepsilon_u^{n+1,h}]) \Delta W_{t_{n+1}}^{\top}] \right\|^2 \\ &\leq \Delta t_n \mathbb{E}_{t_n}^{\zeta} [\|\varepsilon_u^{n+1,h}\|^2] - \Delta t_n \left\| \mathbb{E}_{t_n}^{\zeta} [\varepsilon_u^{n+1,h}] \right\|^2. \end{aligned} \quad (5.26)$$

With $\Delta t \leq C_0 \Delta t_n$ from (5.1), combining (5.24) and (5.25) gives

$$\begin{aligned} & (1 - (1+2L)\Delta t) \mathbb{E} [\|\varepsilon_u^{n,h}\|^2] + \frac{2\nu \Delta t}{C_0} \mathbb{E} [\|\nabla \varepsilon_u^{n,h}\|^2] + \frac{\Delta t}{2C_0} \mathbb{E} [\|\varepsilon_z^{n,h}\|^2] \\ &\leq \mathbb{E} [\|\varepsilon_u^{n+1,h}\|^2] + \Delta t \mathbb{E} [\|\varepsilon_f^{n,h}\|^2]. \end{aligned} \quad (5.27)$$

Taking $\mathbb{E}[\cdot]$ on both sides of (5.27) and summing it over $i = n, n+1, \dots, N-1$ give

$$\begin{aligned} & (1 - (1+2L)\Delta t) \mathbb{E} [\|\varepsilon_u^{N,h}\|^2] + \frac{2\nu}{C_0} \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|\nabla \varepsilon_u^{i,h}\|^2] + \frac{1}{2C_0} \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|\varepsilon_z^{i,h}\|^2] \\ &\leq (1 - (1+2L)\Delta t) \mathbb{E} [\|\varepsilon_u^{N,h}\|^2] + (1+2L)\Delta t \sum_{i=n}^{N-1} \mathbb{E} [\|\varepsilon_u^{i+1,h}\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|\varepsilon_f^{i,h}\|^2]. \end{aligned} \quad (5.28)$$

For proper small $\Delta t > 0$ such that $1 - (1 + 2L)\Delta t > 0$, applying the discrete backward Gronwall's inequality leads to

$$\begin{aligned} & \mathbb{E} [\|\varepsilon_u^{n,h}\|^2] + \sum_{i=n}^{N-1} \nu \Delta t \mathbb{E} [\|\nabla \varepsilon_u^{i,h}\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|\varepsilon_z^{i,h}\|^2] \\ & \leq C \left(\mathbb{E} [\|\varepsilon_u^{N,h}\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|\varepsilon_f^{i,h}\|^2] \right), \end{aligned} \quad (5.29)$$

which completes the proof. $\square$

5.1 Convergence analysis for Scheme 5.1

In this subsection, we devote to give the error estimates of full discrete finite element Scheme 5.1. It is worth to note that in our analyses, we assume that the conditional expectation $\mathbb{E}_{t_n}^{\xi}[\cdot]$ is exactly obtained and do not consider its numerical approximation.

Similar to the error estimates of Scheme 4.1, we define following notations for simplicity:

$$\begin{aligned} e_u^n &= u_{t_n} - u_h^n, & e_p^n &= p_{t_n} - p_h^n, & e_z^n &= z_{t_n} - z_h^n, \\ \rho_u^n &= u_{t_n} - \mathcal{S}_h u_{t_n}, & \theta_u^n &= \mathcal{S}_h u_{t_n} - u_h^n, \\ \rho_p^n &= p_{t_n} - \mathcal{S}_h p_{t_n}, & \theta_p^n &= \mathcal{S}_h p_{t_n} - p_h^n, \\ \rho_z^n &= z_{t_n} - \mathbb{P}_h z_{t_n}, & \theta_z^n &= \mathbb{P}_h z_{t_n} - z_h^n, \\ e_f^n &= f_{t_n} - f_h^n = f(t_n, x, W_{t_n}, y_{t_n}, z_{t_n}) - f(t_n, x, W_{t_n}, y_h^n, z_h^n). \end{aligned} \quad (5.30)$$

Since the estimates of ρ_u^n, ρ_p^n and ρ_z^n are known with the properties of $\mathcal{S}_h$ and $\mathbb{P}_h$ in (3.5) and (3.7), what will be done is deriving the error estimates of θ_u^n, θ_p^n and θ_z^n .

Theorem 5.2. *Under Assumptions 2.1 and 2.2, we assume that $u_h(T) = \mathcal{S}_h u(T)$, $\mathcal{F}_{t_n}$ -measurable $X_h \times M_h \times \mathbb{Z}_h$ -valued (u_h^n, p_h^n, z_h^n) , $0 \leq n \leq N-1$, in Scheme 5.1. Then, for a sufficiently small $\Delta t > 0$, there exists a positive constant C such that*

$$\begin{aligned} & \mathbb{E} [\|\theta_u^n\|^2] + \sum_{i=n}^{N-1} \Delta t \nu \mathbb{E} [\|\nabla \theta_u^i\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|\theta_z^i\|^2] \\ & \leq C \left(\mathbb{E} [\|\theta_u^N\|^2] + \sum_{i=n}^{N-1} \left[\Delta t \left(\mathbb{E} [\|\rho_u^i\|^2] + \mathbb{E} [\|\rho_z^i\|^2] \right) \right. \right. \\ & \quad \left. \left. + \frac{1}{\Delta t} \mathbb{E} [\|\mathbb{E}_{t_i}^{\xi} [\rho_u^{i+1} - \rho_u^i]\|^2] + \frac{1}{\Delta t} \mathbb{E} [\|R_u^i\|^2] \right. \right. \\ & \quad \left. \left. + \frac{1}{\Delta t} \mathbb{E} [\|\mathbb{E}_{t_i}^{\xi} [(\rho_u^{i+1} - \rho_u^i) \Delta W_{t_{i+1}}^{\top}]\|^2] + \frac{1}{\Delta t} \mathbb{E} [\|R_z^i\|^2] \right] \right). \end{aligned} \quad (5.31)$$

Proof. Step 1. Error equations. Letting $v = v_h$ in (5.6), $q = q_h$ in (5.5) and $w = w_h$ in (5.11), subtracting the resulted equations by (5.13), (5.14) and (5.15), respectively, we have the following error equations:

$$\begin{aligned} & (u_{t_n} - u_h^n, v_h) + \nu \Delta t_n (\nabla u_{t_n} - \nabla u_h^n, \nabla v_h) - \Delta t_n (p_{t_n} - p_h^n, \nabla \cdot v_h) \\ &= \left(\mathbb{E}_{t_n}^\xi [u_{t_{n+1}} - u_h^{n+1}], v_h \right) + \Delta t_n (f_{t_n} - f_h^n, v_h) + (R_u^n, v_h), \end{aligned} \quad (5.32)$$

$$(\nabla \cdot u_{t_n} - \nabla \cdot u_h^n, q_h) = 0, \quad (5.33)$$

$$\Delta t_n (z_{t_n} - z_h^n, w_h) = \left(\mathbb{E}_{t_n}^\xi [(u_{t_{n+1}} - u_h^{n+1}) \Delta W_{t_{n+1}}^\top], w_h \right) + (R_z^n, w_h). \quad (5.34)$$

With the property of the projection $\mathcal{S}_h$, we rewrite (5.32) and (5.33) as

$$\begin{aligned} & (\theta_u^n + \rho_u^n, v_h) + \nu \Delta t_n (\nabla \theta_u^n, \nabla v_h) - \Delta t_n (\theta_p^n, \nabla \cdot v_h) \\ &= \left(\mathbb{E}_{t_n}^\xi [\theta_u^{n+1} + \rho_u^{n+1}], v_h \right) + \Delta t_n (e_f^n, v_h) + (R_u^n, v_h), \end{aligned} \quad (5.35)$$

$$(\nabla \cdot \theta_u^n, q_h) = 0, \quad (5.36)$$

$$\Delta t_n (\theta_z^n + \rho_z^n, w_h) = \left(\mathbb{E}_{t_n}^\xi [(\theta_u^{n+1} + \rho_u^{n+1}) \Delta W_{t_{n+1}}^\top], w_h \right) + (R_z^n, w_h). \quad (5.37)$$

Step 2. Estimates of u . Setting $v_h = \theta_u^n$ in (5.35) and $q_h = \theta_p^n$ in (5.36) and considering $(\theta_p^n, \nabla \cdot \theta_u^n) = 0$, we obtain

$$\begin{aligned} & \|\theta_u^n\|^2 + \nu \Delta t_n \|\nabla \theta_u^n\|^2 \\ &= \left(\mathbb{E}_{t_n}^\xi [\rho_u^{n+1} + \theta_u^{n+1}], \theta_u^n \right) + \Delta t_n (e_f^n, \theta_u^n) + (R_u^n, \theta_u^n) - (\rho_u^n, \theta_u^n), \end{aligned} \quad (5.38)$$

which are then estimated by Cauchy-Schwarz inequality as

$$\begin{aligned} & \|\theta_u^n\|^2 + \nu \Delta t_n \|\nabla \theta_u^n\|^2 \\ &\leq \frac{\|\theta_u^n\|^2}{2} + \frac{1}{2} \left\| \mathbb{E}_{t_n}^\xi [\theta_u^{n+1}] + \mathbb{E}_{t_n}^\xi [\rho_u^{n+1} - \rho_u^n] + \Delta t_n e_f^n + R_u^n \right\|^2. \end{aligned} \quad (5.39)$$

By elementary inequality $(a+b)^2 \leq (1+\gamma\Delta t)a^2 + (1+1/(\gamma\Delta t))b^2$ with positive γ and Δt , we estimate the second term at right side of (5.39) as follows:

$$\begin{aligned} & \left\| \mathbb{E}_{t_n}^\xi [\theta_u^{n+1}] + \mathbb{E}_{t_n}^\xi [\rho_u^{n+1} - \rho_u^n] + \Delta t_n e_f^n + R_u^n \right\|^2 \\ &\leq (1+\gamma\Delta t) \left\| \mathbb{E}_{t_n}^\xi [\theta_u^{n+1}] \right\|^2 + \left(1 + \frac{1}{\gamma\Delta t} \right) \left\| \mathbb{E}_{t_n}^\xi [\rho_u^{n+1} - \rho_u^n] + \Delta t_n e_f^n + R_u^n \right\|^2 \\ &\leq (1+\gamma\Delta t) \left\| \mathbb{E}_{t_n}^\xi [\theta_u^{n+1}] \right\|^2 + 3 \left(1 + \frac{1}{\gamma\Delta t} \right) \left(\left\| \mathbb{E}_{t_n}^\xi [\rho_u^{n+1} - \rho_u^n] \right\|^2 + (\Delta t_n)^2 \|e_f^n\|^2 + \|R_u^n\|^2 \right), \end{aligned} \quad (5.40)$$

and we can check that by (4.14)

$$\begin{aligned} \|e_f^n\|^2 &\leq 2L^2 (\|u_{t_n} - u_h^n\|^2 + \|z_{t_n} - z_h^n\|^2) \\ &\leq 4L^2 (\|\rho_u^n\|^2 + \|\theta_u^n\|^2 + \|\rho_z^n\|^2 + \|\theta_z^n\|^2). \end{aligned} \quad (5.41)$$

Collecting (5.39)-(5.41), briefly, we derive that

$$\begin{aligned} & \left(1 - 12 \left(1 + \frac{1}{\gamma \Delta t}\right) L^2 (\Delta t_n)^2\right) \|\theta_u^n\|^2 + 2\Delta t_n \nu \|\nabla \theta_u^n\|^2 \\ & \leq (1 + \gamma \Delta t) \left\| \mathbb{E}_{t_n}^\xi [\theta_u^{n+1}] \right\|^2 + \left(3 + \frac{3}{\gamma \Delta t}\right) \left\| \mathbb{E}_{t_n}^\xi [\rho_u^{n+1} - \rho_u^n] \right\|^2 \\ & \quad + 12 \left(1 + \frac{1}{\gamma \Delta t}\right) L^2 (\Delta t_n)^2 (\|\rho_u^n\|^2 + \|\rho_z^n\|^2 + \|\theta_z^n\|^2) + \left(3 + \frac{3}{\gamma \Delta t}\right) \|R_u^n\|^2. \end{aligned} \quad (5.42)$$

Step 3. Estimates of z . For the estimate of z , similarly, taking $w_h = \theta_z^n$ in (5.37) to get

$$\Delta t_n \|\theta_z^n\|^2 = \left(\mathbb{E}_{t_n}^\xi [(\rho_u^{n+1} + \theta_u^{n+1}) \Delta W_{t_{n+1}}^\top], \theta_z^n \right) + (R_z^n, \theta_z^n) - \Delta t_n (\rho_z^n, \theta_z^n), \quad (5.43)$$

which follows that

$$\begin{aligned} & \left(\mathbb{E}_{t_n}^\xi [(\rho_u^{n+1} + \theta_u^{n+1}) \Delta W_{t_{n+1}}^\top], \theta_z^n \right) + (R_z^n, \theta_z^n) - \Delta t_n (\rho_z^n, \theta_z^n) \\ & \leq \|\theta_z^n\| \left\| \mathbb{E}_{t_n}^\xi [(\rho_u^{n+1} + \theta_u^{n+1}) \Delta W_{t_{n+1}}^\top] + R_z^n \right\| + \Delta t_n \|\rho_z^n\| \|\theta_z^n\|. \end{aligned} \quad (5.44)$$

Then by Cauchy-Schwarz inequality and $ab \leq a^2/(2\epsilon) + \epsilon b^2/2$ with $\epsilon > 0$, we obtain

$$\begin{aligned} & \|\theta_z^n\| \left\| \mathbb{E}_{t_n}^\xi [(\rho_u^{n+1} + \theta_u^{n+1}) \Delta W_{t_{n+1}}^\top] + R_z^n \right\| \\ & \leq \frac{\Delta t_n}{4} \|\theta_z^n\|^2 + \frac{2}{\Delta t_n} \left(\left\| \mathbb{E}_{t_n}^\xi [(\rho_u^{n+1} + \theta_u^{n+1}) \Delta W_{t_{n+1}}^\top] \right\|^2 + \|R_z^n\|^2 \right), \end{aligned} \quad (5.45)$$

$$\Delta t_n \|\rho_z^n\| \|\theta_z^n\| \leq \frac{\Delta t_n}{4} \|\theta_z^n\|^2 + \Delta t_n \|\rho_z^n\|^2. \quad (5.46)$$

According to the martingale property of Brownian motion W_t and

$$\left\| \mathbb{E}_{t_n}^\xi [\theta_u^{n+1} \Delta W_{t_{n+1}}^\top] \right\|^2 \leq \Delta t_n \left(\mathbb{E}_{t_n}^\xi [\|\theta_u^{n+1}\|^2] - \|\mathbb{E}_{t_n}^\xi [\theta_u^{n+1}]\|^2 \right), \quad (5.47)$$

we derive that

$$\begin{aligned} & \left\| \mathbb{E}_{t_n}^\xi [(\theta_u^{n+1} + \rho_u^{n+1}) \Delta W_{t_{n+1}}^\top] \right\|^2 \\ & = \left\| \mathbb{E}_{t_n}^\xi [(\theta_u^{n+1} + \rho_u^{n+1} - \rho_u^n) \Delta W_{t_{n+1}}^\top] \right\|^2 \\ & \leq 2\Delta t_n \left(\mathbb{E}_{t_n}^\xi [\|\theta_u^{n+1}\|^2] - \|\mathbb{E}_{t_n}^\xi [\theta_u^{n+1}]\|^2 \right) + 2 \left\| \mathbb{E}_{t_n}^\xi [(\rho_u^{n+1} - \rho_u^n) \Delta W_{t_{n+1}}^\top] \right\|^2. \end{aligned} \quad (5.48)$$

Combining (5.43), (5.45), (5.46) and (5.48), we have that

$$\begin{aligned} \frac{\Delta t_n}{8} \|\theta_z^n\|^2 & \leq \mathbb{E}_{t_n}^\xi [\|\theta_u^{n+1}\|^2] - \left\| \mathbb{E}_{t_n}^\xi [\theta_u^{n+1}] \right\|^2 + \frac{1}{2\Delta t_n} \|R_z^n\|^2 \\ & \quad + \frac{\Delta t_n}{4} \|\rho_z^n\|^2 + \frac{1}{\Delta t_n} \left\| \mathbb{E}_{t_n}^\xi [(\rho_u^{n+1} - \rho_u^n) \Delta W_{t_{n+1}}^\top] \right\|^2. \end{aligned} \quad (5.49)$$

Multiplying both sides of (5.42) by C_0 , dividing both sides of (5.49) by $\Delta t_n / \Delta t$, and adding the two associated inequalities together, with $\Delta t \leq C_0 \Delta t_n$, we obtain the following estimates as:

$$\begin{aligned}
& C_0 \left(1 - 12 \left(1 + \frac{1}{\gamma \Delta t} \right) L^2 (\Delta t)^2 \right) \|\theta_u^n\|^2 + 2\Delta t \nu \|\nabla \theta_u^n\|^2 \\
& + \left(\frac{\Delta t}{8} - 12 \left(1 + \frac{1}{\gamma \Delta t} \right) C_0 L^2 (\Delta t)^2 \right) \|\theta_z^n\|^2 \\
& \leq C_0 (1 + \gamma \Delta t) \mathbb{E}_{t_n}^\xi [\|\theta_u^{n+1}\|^2] + 12 \left(1 + \frac{1}{\gamma \Delta t} \right) C_0 L^2 (\Delta t)^2 (\|\rho_u^n\|^2 + \|\rho_z^n\|^2) \\
& + 3C_0 \left(1 + \frac{1}{\gamma \Delta t} \right) \left\| \mathbb{E}_{t_n}^\xi [\rho_u^{n+1} - \rho_u^n] \right\|^2 + 3C_0 \left(1 + \frac{1}{\gamma \Delta t} \right) \|R_u^n\|^2 \\
& + \frac{C_0^2}{\Delta t} \left\| \mathbb{E}_{t_n}^\xi [(\rho_u^{n+1} - \rho_u^n) \Delta W_{t_{n+1}}^\top] \right\|^2 + \frac{C_0^2}{2\Delta t} \|R_z^n\|^2 + \frac{\Delta t}{4} \|\rho_z^n\|^2. \tag{5.50}
\end{aligned}$$

We take mathematical expectation $\mathbb{E}[\cdot]$ for (5.50) and let $C_1 = 12(1 + 1/(\gamma \Delta t))L^2 \Delta t$ and $C_2 = 1/8 - C_0 C_1$. (5.50) can be briefly reduced to

$$\begin{aligned}
& C_0 (1 - C_1 \Delta t) \mathbb{E} [\|\theta_u^n\|^2] + 2\Delta t \nu \mathbb{E} [\|\nabla \theta_u^n\|^2] + C_2 \Delta t \mathbb{E} [\|\theta_z^n\|^2] \\
& \leq C_0 (1 + \gamma \Delta t) \mathbb{E} [\|\theta_u^{n+1}\|^2] + C_0 C_1 \Delta t (\mathbb{E} [\|\rho_u^n\|^2] + \mathbb{E} [\|\rho_z^n\|^2]) + \frac{\Delta t}{4} \mathbb{E} [\|\rho_z^n\|^2] \\
& + 3C_0 \left(1 + \frac{1}{\gamma \Delta t} \right) \mathbb{E} \left[\left\| \mathbb{E}_{t_n}^\xi [\rho_u^{n+1} - \rho_u^n] \right\|^2 \right] + 3C_0 \left(1 + \frac{1}{\gamma \Delta t} \right) \mathbb{E} [\|R_u^n\|^2] \\
& + \frac{C_0^2}{\Delta t} \mathbb{E} \left[\left\| \mathbb{E}_{t_n}^\xi [(\rho_u^{n+1} - \rho_u^n) \Delta W_{t_{n+1}}^\top] \right\|^2 \right] + \frac{C_0^2}{2\Delta t} \mathbb{E} [\|R_z^n\|^2]. \tag{5.51}
\end{aligned}$$

Then we choose $\gamma > 0$ large enough, Δt_0 sufficiently small with $\Delta t_0 \geq \Delta t > 0$ such that $C_2 > 0, C_1 \leq C^*, \gamma \leq C^*$ and $1 - C^* \Delta t > 0$, where C^* is a positive constant depending on C_0 and Lipschitz constant L . Summing (5.51) over $i = n, n+1, \dots, N-1$ gives

$$\begin{aligned}
& C_0 (1 - C^* \Delta t) \mathbb{E} [\|\theta_u^n\|^2] + 2 \sum_{i=n}^{N-1} \Delta t \nu \mathbb{E} [\|\nabla \theta_u^i\|^2] + C_2 \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|\theta_z^i\|^2] \\
& \leq C_0 (1 - C^* \Delta t) \mathbb{E} [\|\theta_u^N\|^2] + 2 \sum_{i=n}^{N-1} C_0 C^* \Delta t \mathbb{E} [\|\theta_u^{i+1}\|^2] \\
& + \sum_{i=n}^{N-1} \left(\frac{\Delta t}{4} \mathbb{E} [\|\rho_z^i\|^2] + C_0 C_1 \Delta t (\mathbb{E} [\|\rho_u^i\|^2] + \mathbb{E} [\|\rho_z^i\|^2]) \right. \\
& \quad + 3C_0 \left(1 + \frac{1}{\gamma \Delta t} \right) \mathbb{E} [\left\| \mathbb{E}_{t_i}^\xi [\rho_u^{i+1} - \rho_u^i] \right\|^2] \\
& \quad + \frac{C_0^2}{\Delta t} \mathbb{E} [\left\| \mathbb{E}_{t_i}^\xi [(\rho_u^{i+1} - \rho_u^i) \Delta W_{t_{i+1}}^\top] \right\|^2] \\
& \quad \left. + 3C_0 \left(1 + \frac{1}{\gamma \Delta t} \right) \mathbb{E} [\|R_u^i\|^2] + \frac{C_0^2}{2\Delta t} \mathbb{E} [\|R_z^i\|^2] \right).
\end{aligned}$$

With the discrete backward Gronwall's discrete inequality, it follows that

$$\begin{aligned}
& \mathbb{E}[\|\theta_u^n\|^2] + \sum_{i=n}^{N-1} \Delta t \nu \mathbb{E}[\|\nabla \theta_u^i\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E}[\|\theta_z^i\|^2] \\
& \leq C \left(\mathbb{E}[\|\theta_u^N\|^2] + \sum_{i=n}^{N-1} \left[\Delta t (\mathbb{E}[\|\rho_u^i\|^2] + \mathbb{E}[\|\rho_z^i\|^2]) \right. \right. \\
& \quad \left. \left. + \frac{1}{\Delta t} \mathbb{E}[\|\mathbb{E}_{t_i}^\zeta[\rho_u^{i+1} - \rho_u^i]\|^2] + \frac{1}{\gamma \Delta t} \mathbb{E}[\|R_u^i\|^2] \right. \right. \\
& \quad \left. \left. + \frac{C_0^2}{\Delta t} \mathbb{E}[\|\mathbb{E}_{t_i}^\zeta[(\rho_u^{i+1} - \rho_u^i) \Delta W_{t_{i+1}}^\top]\|^2] + \frac{C_0^2}{2\Delta t} \mathbb{E}[\|R_z^i\|^2] \right] \right), \quad (5.52)
\end{aligned}$$

where $C > 0$ is constant independent of h and Δt . $\square$

Similarly, we proceed to show the following estimates for pressure p in Scheme 5.1.

Theorem 5.3. *Suppose that Assumptions 2.1 and 2.2 hold, there exists a positive constant $C > 0$ such that*

$$\begin{aligned}
\Delta t \sum_{i=n}^{N-1} \mathbb{E}[\|\theta_p^i\|^2] & \leq C \sum_{i=n}^{N-1} \left(\frac{\mathbb{E}[\|\mathbb{E}_{t_i}^\zeta[\rho_u^i - \rho_u^{i+1}]\|^2]}{\Delta t} + \frac{\mathbb{E}[\|\mathbb{E}_{t_i}^\zeta[\theta_u^i - \theta_u^{i+1}]\|^2]}{\Delta t} \right. \\
& \quad \left. + \Delta t \mathbb{E}[\|\nabla \theta_u^i\|^2] + \Delta t \mathbb{E}[\|\rho_u^i\|^2] + \Delta t \mathbb{E}[\|\rho_z^i\|^2] \right. \\
& \quad \left. + \Delta t \mathbb{E}[\|\theta_z^i\|^2] + \frac{\mathbb{E}[\|R_u^i\|^2]}{\Delta t} \right). \quad (5.53)
\end{aligned}$$

Proof. From (5.35) with $v_h \in X_h$, we have

$$\begin{aligned}
\Delta t_n (\theta_p^n, \nabla \cdot v_h) & = (\mathbb{E}_{t_n}^\zeta[e_u^n - e_u^{n+1}], v_h) + \Delta t_n \nu (\nabla \theta_u^n, \nabla v_h) \\
& \quad - \Delta t_n (e_f^n, v_h) - (R_u^n, v_h). \quad (5.54)
\end{aligned}$$

By Cauchy-Schwarz inequality, Poincaré-Friedrichs inequality $\|\theta_u^n\|^2 \leq C \|\nabla \theta_u^n\|^2$ with positive constant C , $\Delta t \leq C_0 \Delta t_n$ and discrete LBB condition (3.1)

$$\sup_{\substack{v_h \in X_h \\ v_h \neq 0}} \frac{(\nabla \cdot v_h, \theta_p^n)}{\|\nabla v_h\|} \geq \bar{C} \|\theta_p^n\|, \quad (5.55)$$

we derive that there is a positive constant C such that

$$\Delta t \|\theta_p^n\| \leq C \left(\|\mathbb{E}_{t_n}^\zeta[e_u^n - e_u^{n+1}]\| + \Delta t \nu \|\nabla \theta_u^n\| + \Delta t \|e_f^n\| + \|R_u^n\| \right). \quad (5.56)$$

Based on triangle inequality, we then get

$$\begin{aligned} \Delta t \|\theta_p^n\| \leq C \Big(& \left\| \mathbb{E}_{t_n}^{\xi} [\rho_u^n - \rho_u^{n+1}] \right\| + \left\| \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}] \right\| \\ & + \Delta t \nu \|\nabla \theta_u^n\| + \Delta t \|e_f^n\| + \|R_u^n\| \Big). \end{aligned} \quad (5.57)$$

We square both sides of above inequality and take $\mathbb{E}[\cdot]$ to obtain

$$\begin{aligned} \Delta t \mathbb{E} [\|\theta_p^n\|^2] \leq C \Big(& \frac{\mathbb{E} [\left\| \mathbb{E}_{t_n}^{\xi} [\rho_u^n - \rho_u^{n+1}] \right\|^2]}{\Delta t} + \frac{\mathbb{E} [\left\| \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}] \right\|^2]}{\Delta t} \\ & + \Delta t \nu^2 \mathbb{E} [\|\nabla \theta_u^n\|^2] + \Delta t \mathbb{E} [\|e_f^n\|^2] + \frac{\mathbb{E} [\|R_u^n\|^2]}{\Delta t} \Big). \end{aligned} \quad (5.58)$$

Then, applying Lipschitz condition of f and Poincaré-Friedrichs inequality and summing the derived inequality over $i = n, n+1, \dots, N-1$ give

$$\begin{aligned} \Delta t \sum_{i=n}^{N-1} \mathbb{E} [\|\theta_p^i\|^2] \leq C \sum_{i=n}^{N-1} \Big(& \frac{\mathbb{E} [\left\| \mathbb{E}_{t_i}^{\xi} [\rho_u^i - \rho_u^{i+1}] \right\|^2]}{\Delta t} + \frac{\mathbb{E} [\left\| \mathbb{E}_{t_i}^{\xi} [\theta_u^i - \theta_u^{i+1}] \right\|^2]}{\Delta t} \\ & + \Delta t \mathbb{E} [\|\nabla \theta_u^i\|^2] + \Delta t \mathbb{E} [\|\rho_u^i\|^2] + \Delta t \mathbb{E} [\|\rho_z^i\|^2] \\ & + \Delta t \mathbb{E} [\|\theta_z^i\|^2] + \frac{\mathbb{E} [\|R_u^i\|^2]}{\Delta t} \Big), \end{aligned} \quad (5.59)$$

which ends the proof. $\square$

For completing the error analysis of Scheme 5.1, we further give the following three lemmas to estimate the corresponding terms in Theorems 5.2 and 5.3.

Lemma 5.1. *Under Assumptions 2.1 and 2.2, there exists a positive constant $C > 0$ such that*

$$\begin{aligned} & \sum_{i=n}^{N-1} \frac{\mathbb{E} [\left\| \mathbb{E}_{t_i}^{\xi} [\theta_u^i - \theta_u^{i+1}] \right\|^2]}{\Delta t} \\ \leq C \sum_{i=n}^{N-1} \Big(& \frac{\mathbb{E} [\left\| \mathbb{E}_{t_i}^{\xi} [\rho_u^{i+1} - \rho_u^i] \right\|^2]}{\Delta t} + \frac{\mathbb{E} [\|R_u^i\|^2]}{\Delta t} \\ & + \Delta t \Big(\mathbb{E} [\|\rho_u^i\|^2] + \left(1 + \frac{\nu^2}{h^2}\right) \mathbb{E} [\|\nabla \theta_u^i\|^2] + \mathbb{E} [\|\rho_z^i\|^2] + \mathbb{E} [\|\theta_z^i\|^2] \Big) \Big). \end{aligned} \quad (5.60)$$

Proof. We set $v_h = \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}]$ in (5.35) and note that

$$\left(\nabla \cdot \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}], q_h \right) = 0, \quad \forall q_h \in M_h, \quad (5.61)$$

from which we have

$$\begin{aligned} \left\| \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}] \right\|^2 &= \nu \Delta t_n \left(\nabla \theta_u^n, \nabla \mathbb{E}_{t_n}^{\xi} [\theta_u^{n+1} - \theta_u^n] \right) + \left(\mathbb{E}_{t_n}^{\xi} [\rho_u^{n+1} - \rho_u^n], \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}] \right) \\ &\quad + \Delta t_n \left(e_f^n, \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}] \right) + \left(R_u^n, \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}] \right). \end{aligned} \quad (5.62)$$

By Cauchy-Schwarz inequality and inverse inequality $\|v_h\|_1 \leq C\|v_h\|/h$, we obtain that

$$\begin{aligned} &\nu \Delta t_n \left(\nabla \theta_u^n, \nabla \mathbb{E}_{t_n}^{\xi} [\theta_u^{n+1} - \theta_u^n] \right) \\ &\leq \nu \Delta t_n \|\nabla \theta_u^n\| \left\| \nabla \mathbb{E}_{t_n}^{\xi} [\theta_u^{n+1} - \theta_u^n] \right\| \\ &\leq \frac{\nu \Delta t_n}{h} \|\nabla \theta_u^n\| \left\| \mathbb{E}_{t_n}^{\xi} [\theta_u^{n+1} - \theta_u^n] \right\| \\ &\leq \frac{1}{4} \left\| \mathbb{E}_{t_n}^{\xi} [\theta_u^{n+1} - \theta_u^n] \right\|^2 + \frac{C\nu^2(\Delta t_n)^2}{h^2} \|\nabla \theta_u^n\|^2, \end{aligned} \quad (5.63)$$

and

$$\begin{aligned} &\left(\mathbb{E}_{t_n}^{\xi} [\rho_u^{n+1} - \rho_u^n] + \Delta t_n e_f^n + R_u^n, \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}] \right) \\ &\leq \frac{1}{4} \left\| \mathbb{E}_{t_n}^{\xi} [\theta_u^{n+1} - \theta_u^n] \right\|^2 + C \left(\left\| \mathbb{E}_{t_n}^{\xi} [\rho_u^{n+1} - \rho_u^n] \right\|^2 + \|R_u^n\|^2 + (\Delta t_n)^2 \|e_f^n\|^2 \right). \end{aligned} \quad (5.64)$$

Combining (5.62)-(5.64), we can get the following estimates:

$$\begin{aligned} &\left\| \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}] \right\|^2 \\ &\leq C \left(\frac{\nu^2(\Delta t_n)^2}{h^2} \|\nabla \theta_u^n\|^2 + \left\| \mathbb{E}_{t_n}^{\xi} [\rho_u^{n+1} - \rho_u^n] \right\|^2 + \|R_u^n\|^2 + (\Delta t_n)^2 \|e_f^n\|^2 \right). \end{aligned} \quad (5.65)$$

Taking $\mathbb{E}[\cdot]$ in above (5.65), by Lipschitz condition (4.14) and Poincaré-Friedrichs inequality, it follows that

$$\begin{aligned} &\mathbb{E} \left[\left\| \mathbb{E}_{t_n}^{\xi} [\theta_u^n - \theta_u^{n+1}] \right\|^2 \right] \\ &\leq C \left(\mathbb{E} \left[\left\| \mathbb{E}_{t_n}^{\xi} [\rho_u^{n+1} - \rho_u^n] \right\|^2 \right] + \mathbb{E} [\|R_u^n\|^2] \right. \\ &\quad \left. + (\Delta t)^2 \left(\mathbb{E} [\|\rho_u^n\|^2] + \left(1 + \frac{\nu^2}{h^2} \right) \mathbb{E} [\|\nabla \theta_u^n\|^2] + \mathbb{E} [\|\rho_z^n\|^2] + \mathbb{E} [\|\theta_z^n\|^2] \right) \right). \end{aligned} \quad (5.66)$$

Summing (5.66) over $i = n, n+1, \dots, N-1$ gives

$$\begin{aligned} & \sum_{i=n}^{N-1} \mathbb{E} \left[\left\| \mathbb{E}_{t_i}^{\zeta} [\theta_u^i - \theta_u^{i+1}] \right\|^2 \right] \\ & \leq C \sum_{i=n}^{N-1} \left(\mathbb{E} \left[\left\| \mathbb{E}_{t_i}^{\zeta} [\rho_u^{i+1} - \rho_u^i] \right\|^2 \right] + \mathbb{E} [\|R_u^i\|^2] \right. \\ & \quad \left. + (\Delta t)^2 \left(\mathbb{E} [\|\rho_u^i\|^2] + \left(1 + \frac{\nu^2}{h^2}\right) \mathbb{E} [\|\nabla \theta_u^i\|^2] + \mathbb{E} [\|\rho_z^i\|^2] + \mathbb{E} [\|\theta_z^i\|^2] \right) \right), \end{aligned} \quad (5.67)$$

which completes the proof. $\square$

Lemma 5.2 ([43]). *Let R_u^n and R_z^n be the truncation errors defined in (5.7)-(5.9) and (5.12). If f and terminal condition φ are smooth and bounded with their derivatives, for sufficiently small time step $\Delta t_n > 0, n = 1, 2, \dots, N-1$, there exists a positive constant C such that*

$$\|R_u^n\|^2 + \|R_z^n\|^2 \leq C(\Delta t_n)^4. \quad (5.68)$$

The main result of error estimates for the fully discrete Scheme 5.1 is presented in the following theorem.

Theorem 5.4. *With conditions in Theorems 5.2 and 5.2, suppose that Assumptions 2.1 and 2.2 hold. We set $u_h^N = S_h u_T$ and let*

$$\begin{aligned} (u, p, z) \in & \left(C_{\mathcal{F}}([0, T]; L^2(\Omega, X \cap (H^{k+4}(\mathcal{D}))^d)) \cap L_{\mathcal{F}}^2([0, T]; X \cap (H^{k+4}(\mathcal{D}))^d) \right) \\ & \times L_{\mathcal{F}}^2([0, T]; M \cap H^{k+3}(\mathcal{D})) \times L_{\mathcal{F}}^2([0, T]; Z \cap (H^{k+2}(\mathcal{D}))^{d \times q}). \end{aligned}$$

Then for $0 \leq n \leq N-1, k \geq 1$, there exist a sufficiently small $\Delta t > 0$ and constant $C > 0$ such that the following error estimates hold:

$$\begin{aligned} & \mathbb{E} [\|e_u^n\|^2] + h^2 \sum_{i=n}^{N-1} \Delta t \nu \mathbb{E} [\|\nabla e_u^i\|^2] + h^2 \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|e_p^i\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|e_z^i\|^2] \\ & \leq C(h^{2(k+2)} + (\Delta t)^2). \end{aligned} \quad (5.69)$$

Proof. According to the definitions of the errors e_u^n, e_p^n and e_z^n in (5.30), considering Theorems 5.2, 5.3 and Lemma 5.1, and using the triangle inequalities, we have

$$\begin{aligned} & \mathbb{E} [\|e_u^n\|^2] + h^2 \sum_{i=n}^{N-1} \Delta t \nu \mathbb{E} [\|\nabla e_u^i\|^2] + h^2 \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|e_p^i\|^2] + \sum_{i=n}^{N-1} \Delta t \mathbb{E} [\|e_z^i\|^2] \\ & \leq 2 \left[\mathbb{E} [\|\theta_u^n\|^2] + \mathbb{E} [\|\rho_u^n\|^2] + h^2 \sum_{i=n}^{N-1} \Delta t \nu \left(\mathbb{E} [\|\nabla \theta_u^i\|^2] + \mathbb{E} [\|\nabla \rho_u^i\|^2] \right) \right] \end{aligned}$$

$$\begin{aligned}
& + h^2 \sum_{i=n}^{N-1} \Delta t \left(\mathbb{E} [\|\theta_p^i\|^2] + \mathbb{E} [\|\rho_p^i\|^2] \right) + \sum_{i=n}^{N-1} \Delta t \left(\mathbb{E} [\|\theta_z^i\|^2] + \mathbb{E} [\|\rho_z^i\|^2] \right) \\
\leq & C \left(\sum_{i=n}^{N-1} \left[\Delta t \left(h^2 \mathbb{E} [\|\nabla \rho_u^i\|^2] + h^2 \mathbb{E} [\|\rho_p^i\|^2] + \mathbb{E} [\|\rho_u^i\|^2] + \mathbb{E} [\|\rho_z^i\|^2] \right) \right. \right. \\
& + \sum_{i=n}^{N-1} \frac{1}{\Delta t} \left(\mathbb{E} [\|\mathbb{E}_{t_i}^{\xi} [\rho_u^{i+1} - \rho_u^i]\|^2] + \mathbb{E} [\|\mathbb{E}_{t_i}^{\xi} [(\rho_u^{i+1} - \rho_u^i) \Delta W_{t_{i+1}}^{\top}]\|^2] \right. \\
& \quad \left. \left. + h^2 \mathbb{E} [\|\mathbb{E}_{t_i}^{\xi} [\theta_u^i - \theta_u^{i+1}]\|^2] + \mathbb{E} [\|R_u^i\|^2] + \mathbb{E} [\|R_z^i\|^2] \right) \right. \\
& \quad \left. \left. + \mathbb{E} [\|\theta_u^N\|^2] + h^2 \mathbb{E} [\|\nabla \theta_u^N\|^2] \right) \right). \tag{5.70}
\end{aligned}$$

$$\|e_u^n\|^2 + \|e_z^n\|^2 \leq 2(\|\theta_u^n\|^2 + \|\rho_u^n\|^2 + \|\theta_z^n\|^2 + \|\rho_z^n\|^2), \tag{5.71}$$

$$h^2(\|\nabla e_u^n\|^2 + \|e_p^n\|^2) \leq 2h^2(\|\nabla \theta_u^n\|^2 + \|\nabla \rho_u^n\|^2 + \|\theta_p^n\|^2 + \|\rho_p^n\|^2). \tag{5.72}$$

With the estimations of $\mathbb{E} [\|\mathbb{E}_{t_i}^{\xi} [\rho_u^{i+1} - \rho_u^i]\|^2]$ and $\mathbb{E} [\|\mathbb{E}_{t_i}^{\xi} [(\rho_u^{i+1} - \rho_u^i) \Delta W_{t_{i+1}}^{\top}]\|^2]$ in Lemma 2.4, the truncation errors $\mathbb{E} [\|R_u^i\|^2]$ and $\mathbb{E} [\|R_z^i\|^2]$ in Lemma 5.2, we establish this theorem by the properties of operators $\mathcal{S}_h$ and $\mathbb{P}_h$ in (3.5) and (3.7). $\square$

6 Numerical tests

Numerical experiments below are designed to validate our theoretical results in Section 5.1. All the computations are parallelly performed on HPC with 512 Gb memory in random space. The resulted nonlinear discrete systems are solved by Newton methods with MUMPS solver [1]. To perform the finite element Scheme 5.1, the computation of $\mathbb{E}_{t_n}^{\xi} [\cdot]$ is required. In this paper, we refer to [37, 42] for the detailed implementations.

Example 6.1. Consider the Taylor-Green vortex flows in [41] and reformulate them into stochastic processes with 1- d standard Brownian motion W_t as

$$\begin{aligned}
u^T &= (u_1, u_2) = e^{-2\nu t} (\sin(\pi x + W_t) \cos(\pi y + W_t), -\cos(\pi x + W_t) \sin(\pi y + W_t)), \\
z^T &= (z_1, z_2) = e^{-2\nu t} (\cos(\pi x + \pi y + 2W_t), -\cos(\pi x + \pi y + 2W_t)), \\
p &= \frac{1}{4} e^{-4\nu t} (\cos(2\pi x + W_t) + \sin(2\pi y + W_t)).
\end{aligned}$$

According to Feynman-Kac and Itô formulas, one can check that (u, p, z) is a solution of BSSEs defined in (1.1) with $f = (f_1, f_2)^{\top}$,

$$\begin{aligned}
f_1 &= 2\nu(1+\pi^2)u_1 + z_1 - \frac{\pi}{2}\exp(-4\nu t)\sin(2\pi x + W_t) \\
&\quad + \exp(-2\nu t)(\sin(\pi x + \pi y + 2W_t) - \cos(\pi x + \pi y + 2W_t)), \\
f_2 &= 2\nu(1+\pi^2) \cdot u_2 + z_2 + \frac{\pi}{2}\exp(-4\nu t)\sin(2\pi x + W_t) \\
&\quad - \exp(-2\nu t)(\sin(\pi x + \pi y + 2W_t) - \cos(\pi x + \pi y + 2W_t)).
\end{aligned}$$

The boundary and terminal conditions of (1.1) are then followed from above.

In the numerical experiments, the spatial triangulations are uniformly constructed by partitioning the square domain $\mathcal{D} = [0, 1] \times [0, 1]$ into $n \times n$ uniform subrectangles and then dividing each square element into two triangles along the diagonal lines. The approximation of conditional $\mathbb{E}_{t_n}^{\xi}[\cdot]$ are done by referring to [36, 37, 42]. Moreover, the Taylor-Hood finite element pair (P_2, P_1) is used throughout our computations. To be brief, let $\|e_u^0\|$, $\|e_u^0\|_1$, $\|e_p^0\|$, $\|e_p^0\|_1$, $\|e_z^0\|$ and $\|e_z^0\|_1$ represent $\|u_0 - u_h^0\|$, $\|u_0 - u_h^0\|_1$, $\|p_0 - p_h^0\|$, $\|p_0 - p_h^0\|_1$, $\|z_0 - z_h^0\|$ and $\|z_0 - z_h^0\|_1$, respectively, which are the errors between the exact solution (u_t, p_t, z_t) in (1.1) at time $t = 0$ and the numerical solution (u_h^t, p_h^t, z_h^t) of Scheme 5.1 at $n = 0$. To do the convergence simulation, we use 32 CPU processors. Statistically, it takes approximate 88644 seconds to complete convergence tests.

The computational results of the errors and convergence rates are listed in Tables 1 and 2. For the time convergence tests, we set $T = 1$ and choose $\Delta t = 1/20, 1/25, 1/30, 1/35$ as the time steps with a fixed space mesh size $h = 1/2^6$. Table 1 shows that the order of convergence rates in time for all u, p, z in L^2 and H^1 norms is 1. The spatial convergence results are presented in Table 2, which are calculated with mesh sizes $h = 1/4, 1/8, 1/16$ and $1/32$ and a fixed $\Delta t = 1/10^5$. By choosing the Taylor-Hood element (P_2, P_1) , the opti-

Table 1: Errors and time convergence rates in time for Example 6.1.

$1/\Delta t$	$\ e_u^0\ $	$\ e_u^0\ _1$	$\ e_p^0\ $	$\ e_p^0\ _1$	$\ e_z^0\ $	$\ e_z^0\ _1$
20	4.4426E-03	3.3578E-02	5.5061E-02	2.6255E-01	1.8021E-01	7.8910E+00
25	3.5956E-03	2.7184E-02	4.4877E-02	2.1404E-01	1.4698E-01	6.4364E+00
30	3.0199E-03	2.2836E-02	3.7867E-02	1.8081E-01	1.2407E-01	5.4339E+00
35	2.6028E-03	1.9685E-02	3.2749E-02	1.5665E-01	1.0734E-01	4.7013E+00
TCR	0.9552	0.9541	0.9283	0.9228	0.9256	0.9252

Table 2: Errors and space convergence rates in space for Example 6.1.

$1/h$	$\ e_u^0\ $	$\ e_u^0\ _1$	$\ e_p^0\ $	$\ e_p^0\ _1$	$\ e_z^0\ $	$\ e_z^0\ _1$
4	5.9750E-03	1.8494E-01	9.9533E-02	7.4753E-01	1.1441E-02	3.5481E-01
8	7.6287E-04	4.7294E-02	1.6165E-02	3.6162E-01	1.4787E-03	9.1021E-02
16	9.6761E-05	1.1910E-02	3.6631E-03	1.7878E-01	1.8863E-04	2.2946E-02
32	1.2161E-05	2.9835E-03	8.9473E-04	8.9131E-02	2.4192E-05	5.7500E-03
SCR	2.9800	1.9851	2.2534	1.0220	2.9627	1.9829

mal orders of spatial convergence rates 3 (respectively 2) in L^2 norm and 2 (respectively 1) in H^1 norm for u, z (respectively p), respectively, are obtained. The computational results shown in Tables 1 and 2 are all consistent with our theoretical result in Theorem 5.4.

Example 6.2. Referring to [15], we extend the forward stochastic case into the backward stochastic case as

$$\begin{aligned} u^T &= (u_1, u_2) = \left(\phi(x, y) \sin\left(W_t + \frac{\pi}{4}\right), \psi(x, y) \sin\left(W_t + \frac{\pi}{4}\right) \right), \\ z^T &= (z_1, z_2) = \left(\phi(x, y) \cos\left(W_t + \frac{\pi}{4}\right), \psi(x, y) \cos\left(W_t + \frac{\pi}{4}\right) \right), \\ p &= \frac{3}{5} \left(x^3 y^2 + xy + x + y - \frac{4}{3} \right) \sin\left(W_t + \frac{\pi}{4}\right) \end{aligned}$$

with

$$\begin{aligned} \phi(x, y) &= (x + x^2 + x^3 - 2xy - 3xy^2 + x^2y), \\ \psi(x, y) &= (-y + y^2 + y^3 - 2xy - 3x^2y - xy^2). \end{aligned}$$

The $f = (f_1, f_2)^\top$ is induced by Feynman-Kac formula and Itô formula,

$$\begin{aligned} f_1 &= \frac{1}{2} u_1 - u_1 \cos\left(W_t + \frac{\pi}{4}\right) + z_1 \sin\left(W_t + \frac{\pi}{4}\right) \\ &\quad + \frac{3}{5} (1 + y + 3x^2 y^2) \sin\left(W_t + \frac{\pi}{4}\right) - 2\nu \sin\left(W_t + \frac{\pi}{4}\right) (y + 1), \\ f_2 &= \frac{1}{2} u_2 + u_2 \cos\left(W_t + \frac{\pi}{4}\right) - z_2 \sin\left(W_t + \frac{\pi}{4}\right) \\ &\quad + \frac{3}{5} (1 + x + 2x^3 y) \sin\left(W_t + \frac{\pi}{4}\right) + 2\nu \sin\left(W_t + \frac{\pi}{4}\right) (x - 1). \end{aligned}$$

All the configurations are the same as those for Example 6.1. Tables 3 and 4 present computational results with respect to time and space, respectively. Table 3 illustrates that the first-order convergence rates are all obtained in time for u, p, z in L^2 and H^1 -norms. Table 4 confirms that the space convergence rates is 3 (respectively 2) and 2 (respectively 1) for u, z (respectively p) with L^2 and H^1 -norms are also achieved. All the numerical tests demonstrate the effectiveness of the proposed numerical Scheme 5.1 and verify the correction of our theoretical results.

Table 3: Errors and time convergence rates in time for Example 6.2.

$1/\Delta t$	$\ e_u^0\ $	$\ e_u^0\ _1$	$\ e_p^0\ $	$\ e_p^0\ _1$	$\ e_z^0\ $	$\ e_z^0\ _1$
20	1.6547E-04	1.2967E-03	7.8401E-03	2.6323E-02	3.0842E-02	1.7724E+00
25	1.3266E-04	1.0398E-03	6.2792E-03	2.1365E-02	2.4735E-02	1.4214E+00
30	1.1071E-04	8.6798E-04	5.2347E-03	1.8093E-02	2.0647E-02	1.1865E+00
35	9.4989E-05	7.4491E-04	4.4861E-03	1.5784E-02	1.7718E-02	1.0182E+00
TCR	0.9918	0.9906	0.9975	0.9146	0.9904	0.9904

Table 4: Errors and space convergence rates in space for Example 6.2.

$1/h$	$\ e_u^0\ $	$\ e_u^0\ _1$	$\ e_p^0\ $	$\ e_p^0\ _1$	$\ e_z^0\ $	$\ e_z^0\ _1$
4	1.0605E-03	3.3816E-02	6.5832E-02	3.4362E-01	1.0606E-03	3.3816E-02
8	1.2602E-04	8.2897E-03	1.4822E-02	1.1271E-01	1.2608E-04	8.2900E-03
16	1.5489E-05	2.0622E-03	3.5229E-03	4.7809E-02	1.5526E-05	2.0623E-03
32	1.9263E-06	5.1490E-04	8.7753E-04	2.2743E-02	2.0356E-06	5.1529E-04
SCR	3.0338	2.0118	2.0760	1.2989	3.0097	2.0115

Example 6.3. The lid-driven cavity flows on $\mathcal{D} = [0,1] \times [0,1]$ are considered in this example, for which we set the Dirichlet boundary condition with $\alpha, \beta \in \mathbb{R}$,

$$u(t, x, y, W_t) = \begin{cases} (\alpha + \beta \sin(W_t), 0), & y = 1, \quad 0 < x < 1, \\ (0, 0), & \text{otherwise,} \end{cases} \quad (6.1)$$

and terminal condition $u(T, x, y, W_T)$. In the following tests, the parameters are taken as $\Delta t = 1/100, \nu = 0.1, T = 1$ and the square domain $\mathcal{D}$ is regularly partitioned into 6312 triangle elements. To explain the simulation results, in the following plots, we consider the 1- d Wiener process with five paths $\omega_1, \dots, \omega_5$ as shown in Fig. 1. To capture all the scenarios, a terabyte of data is approximately stored in each simulation. Since it is impossible to present all the scenarios of u_h^n , in order to provide the evolutionary states of the

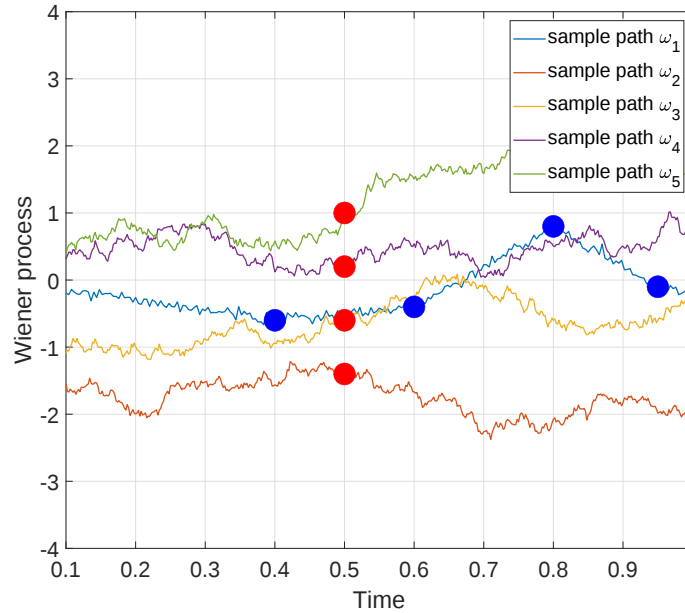


Figure 1: Wiener process with 5 paths.

backward stochastic processes u_h^n and z_h^n , we choose the points $(t_n, W_{t_n}(\omega))$ with $\omega = \omega_1$ at $t_n = 0.4, 0.6, 0.8, 0.98$, and mark them with blue dots in Fig. 1. On the other hand, for comparing the scenarios of different paths, we also consider the points $(t_n, W_{t_n}(\omega))$ along different paths $\omega = \omega_2, \dots, \omega_5$ at $t_n = 0.5$, and mark them with red dots in Fig. 1.

As it is known [32], when the terminal and boundary conditions in model (1.1) are both deterministic, the unknown stochastic process z in model (1.1) vanishes. In the case of model with $\alpha=1, \beta=0$, the backward stochastic Stokes equations (1.1) are equivalent to the backward deterministic Stokes equations, which implies that $\mathbb{E}_{t_n}^{\zeta}[u_h^n]$ are all the same with different realizations $\zeta \in \mathbb{R}$ and $\mathbb{E}_{t_n}^{\zeta}[z_h^n] = 0$. Based on this, Fig. 2 displays the magnitudes and streamlines of velocity u_h^n along path ω_1 at time $t_n = 0.4, 0.6, 0.8, 0.98$. Similarly to the deterministic counterpart, it can be seen that with time-reversed evolution two Moffat eddies at the lower left and right corners of the cavity are also gradually appeared in Fig. 2.

The backward stochastic lid-driven cavity flows with stochastic final and boundary conditions are further considered with $\alpha=0, \beta=1$. To describe the time-reversed evolution of processes u_h^n and z_h^n , the magnitudes and streamlines of u_h^n and z_h^n along path ω_1 at $t = 0.4, 0.6, 0.8, 0.98$ are presented in Fig. 3. With the time evolution in reverse direction, two Moffat eddies at the bottom corners are gradually generated. We also see that there are two Moffat eddies appearing for process z_h^n from Fig. 3. The flow direction of u_h^n changes over time, which is different from Fig. 2. For the purpose of comparison, the computational results of u_h^n and z_h^n along paths $\omega_2, \dots, \omega_5$ at $t = 0.5$ are depicted in Fig. 4, which describe the magnitudes and flow directions of u_h^n and z_h^n are different for different paths. Figs. 3 and 4 show that the flow directions of u_h^n and z_h^n appear clockwise or counter-clockwise orientation in different paths. The reason is that u_h^n and z_h^n are both stochastic processes. According to Feynman-Kac formula, z_h^n depends on the rate of change of u_h^n with respect to W_t , which makes the flow direction of z_h^n be different or same with u_h^n . The above numerical tests demonstrate that the accuracy and effectiveness of proposed Scheme 5.1 for solving the nonlinear BSSEs (1.1)-(1.2). All the computational simulations are consistent with our theoretical results.

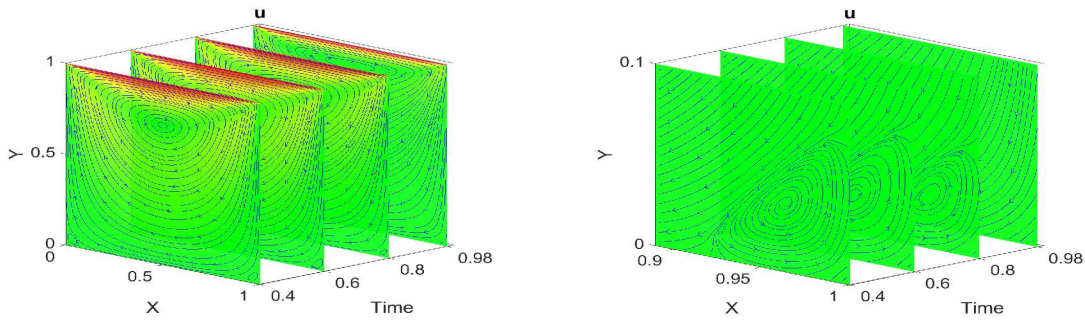


Figure 2: Process u_h^n along path ω_1 with $(\alpha, \beta) = (1, 0)$. Streamlines of velocities (left). Zoom in plots of right bottom corners (right).

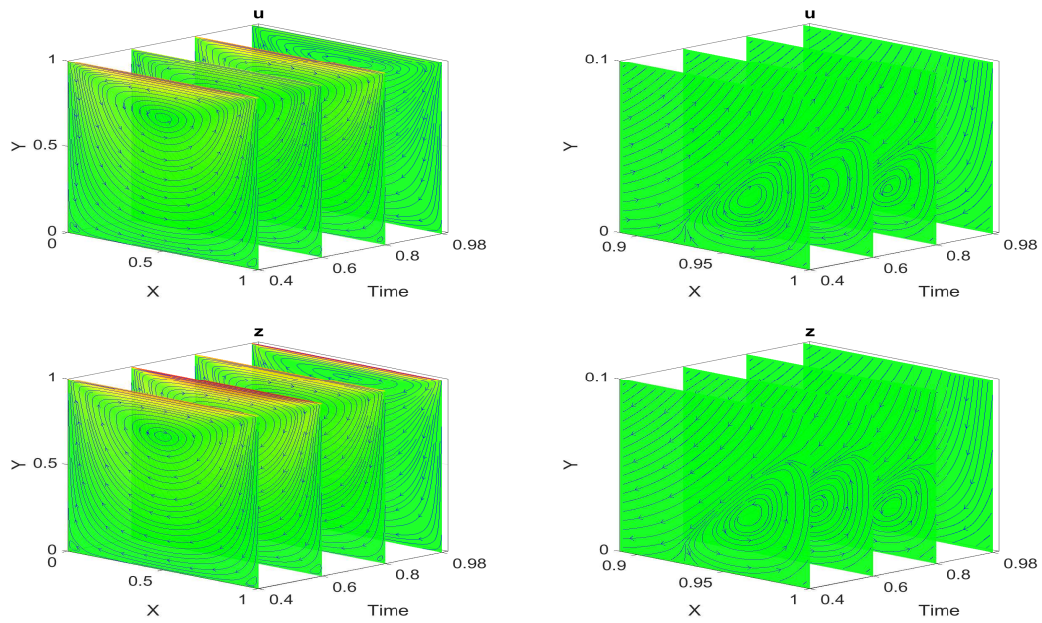


Figure 3: Processes u_h^n, z_h^n along path ω_1 with $(\alpha, \beta) = (0, 1)$. Streamlines of velocities (left). Zoom in plots of right bottom corners (right).

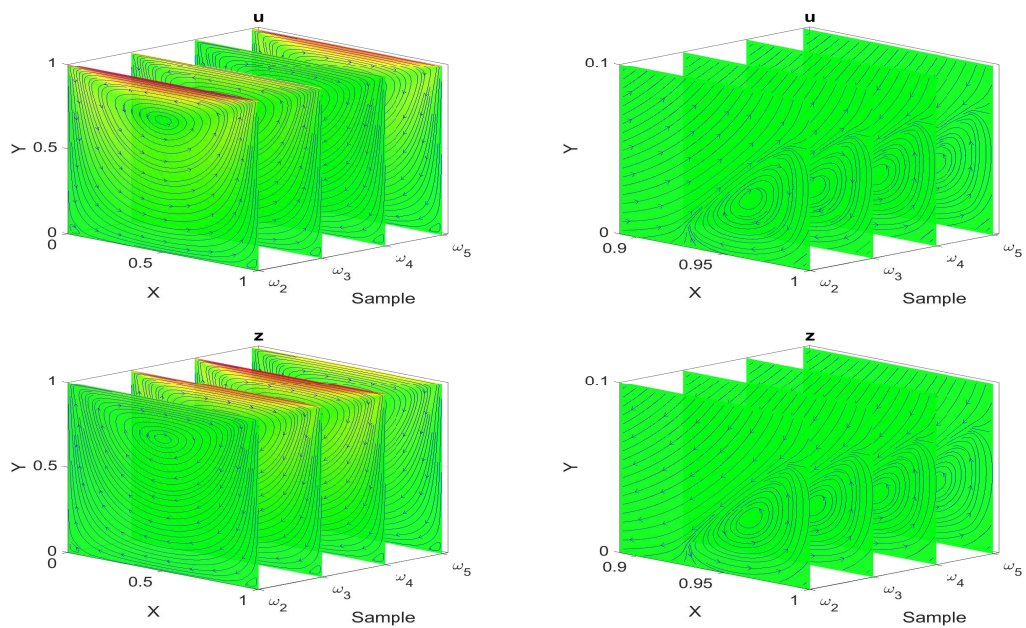


Figure 4: Processes u_h^n, z_h^n along path $\omega_2, \omega_3, \omega_4, \omega_5$ with $(\alpha, \beta) = (0, 1)$ at $t = 0.5$. Streamlines of velocities (left). Zoom in plots of right bottom corners (right).

7 Conclusions

In this paper, we propose a fully discrete finite element scheme for solving backward stochastic Stokes equations and rigorously derived their error estimates. We prove that this scheme has optimal spatial convergence rate and first-order convergence rate in time. The numerical results demonstrate the effectiveness of proposed our scheme and are consistent with the theoretical analysis. The numerical techniques in this paper can be applied to more complex models.

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