CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2024-0005

Vol. **6**, No. 1, pp. 63-95
March 2025

A Variational Discretization Method for Mean Curvature Flows by the Onsager Principle

Yihe Liu and Xianmin Xu*

LSEC, ICMSEC, NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China

Received 1 February 2024; Accepted 27 October 2024

Abstract. The mean curvature flow describes the evolution of a surface (a curve) with normal velocity proportional to the local mean curvature. It has many applications in mathematics, science and engineering. In this paper, we develop a numerical method for mean curvature flows by using the Onsager principle as an approximation tool. We first show that the mean curvature flow can be derived naturally from the Onsager variational principle. Then we consider a piecewise linear approximation of the curve and derive a discrete geometric flow. The discrete flow is described by a system of ordinary differential equations for the nodes of the discrete curve. We prove that the discrete system preserve the energy dissipation structure in the framework of the Onsager principle and this implies the energy decreasing property. The ODE system can be solved by the improved Euler scheme and this leads to an efficient fully discrete scheme. We first consider the method for a simple mean curvature flow and then extend it to the volume preserving mean curvature flow and also a wetting problem on substrates. Numerical examples show that the method has optimal convergence rate and works well for all the three problems.

AMS subject classifications: 53E10, 82C35, 65M60

Key words: Mean curvature flow, the Onsager principle, moving finite element method.

1 Introduction

Mean curvature flow describes the process of surface evolution, where a surface moves in its normal direction with a velocity equal to its mean curvature, i.e.

$$\vec{v} = -H\vec{n}, \tag{1.1}$$

where $\vec{v}$ denotes the velocity of motion of the surface, and H denotes the mean curvature of the surface. $\vec{n}$ is the unit normal vector of the surface. The mean curvature flow

*Corresponding author. *Email addresses:* `liuyihe@amss.ac.cn` (Y. Liu), `xmxu@lsec.cc.ac.cn` (X. Xu)

first appeared in materials science in the 1920s, and the general mathematical model was first proposed in 1957 in [40], where Mullins used it to describe how grooves develop on the surfaces of hot polycrystals. The mean curvature flow has found numerous applications. One is in image smoothing, where it is used to improve image quality by removing unnecessary noise and enhancing specific features without altering the transmitted information [27]. Additionally, the mean curvature flow has been employed to model a special form of the reaction-diffusion equation in chemical reactions [45].

The numerical approximation of mean curvature flow originates from the pioneering work of Dziuk [19] in 1990. He proposed a parametric finite element method (PFEM) to solve the problem based on the observation that the mean curvature flow is a diffusion equation for the surface embedding in a bulk region. However, as the surface evolves, the nodes in the PFEMs may overlap, leading to mesh distortion [28]. Subsequently, various techniques have been introduced in PFEMs that allow points on interpolated curves or surfaces to shift tangentially to produce a better distribution of nodes. For example, Deckelnick and Dziuk [14] introduced an artificial tangential velocity. By similar motivation, Elliott and Fritz [22] employed a DeTurck's trick to introduce a tangential velocity. A widely used method is the parametric finite element method developed by Barrett *et al.* [5–7]. They utilize a different variational form which can induce tangential velocity for the mesh nodes automatically. The method have been generalized some other geometrical flows [2–4, 8–10, 28, 29, 36]. Recently, there are significant progresses in rigorous convergence analysis for discrete schemes for mean curvature flows and some other geometric flows [1, 31–34]. In addition, there exist many other methods to approximate the mean curvature flow numerically, such as the threshold dynamics method [23, 39, 46, 49, 51], the level set method [12, 25, 44, 47, 48, 53], and the phase field method [20, 21, 24, 26], etc.

In this paper, we develop a new finite element method for mean curvature flow by using the Onsager principle as an approximation tool. We consider three different problems, namely the standard mean curvature flow, the volume preserving mean curvature flow and a wetting problem with contact with substrate. We show that the dynamic equations of all the three problems can be derived naturally by the Onsager principle. Furthermore, by considering a piecewise linear approximation of the surface, we can derive semi-discrete problems which preserve the energy dissipation relations just as the continuous problems. The discrete volume can also be preserved if the continuous problem preserve the volume. The semi-discrete problems can be further discretized in time by the improved Euler scheme. Numerical examples show that the method works well for all the three problems. In particular, it has optimal convergence rate in space for a mean curvature flow with analytic solutions. We mainly consider mean curvature flows of a curve in two dimensional space, also known as curve shortening flow in literature. Our method can be generalized to higher dimensional problems while there will be more restrictions on the time discrete schemes to avoid mesh distortion.

The structure of the paper is as follows. In Section 2, we introduce the main idea to use the Onsager variational principle as an approximation tool. In Section 3, we first