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Economic ProductRank and Quantum Wave Probability

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Abstract. This note discusses a long debated question whether there is randomness in quantum mechanics or not? Einstein's view on the question is "God does not throw dice". Our starting point for the discussion is the classification of products in the economic system, called ProductRank, which seems an analog of the "principal component analysis" in statistics. But the former is much more elaborate than the latter. Interestingly, we find an intrinsic common point among economic system, statistics and quantum mechanics, which then leads to a successful classification of the products in economy, as well as a mathematical view of "wave probability" in quantum mechanics. An application to the algorithm for eigenpair is included.

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1 Introduction

The problem mentioned above was motivated from Born's suggestion (1926) saying that the Schrödinger's wave function describes "waves of probability": "The square of the amplitude (of the wave function) represents the probability density of finding the particle in a certain place at a certain time"(cf. [9, p. 114]). Refer also to [4] for a survey and references therein. Here we introduce a mathematical view of Born's annotation based on our recent study on the classification of the products in economic system. For which an advanced probabilistic tool – Markov chains is adopted. However, as can be seen very soon that there is essentially no randomness in the story.

The main results of the note are stated in the next two sections. First on economics (Section 2), then on quantum, and finally on algorithm (Section 3). Their proofs are delayed to the last section (Section 4).

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2 Ranking the products in an economic system

Denote by $x=(x^{(1)},x^{(2)},\ldots,x^{(d)})$ the vector of products we are interested in the economic system. Then the evolution of the system is mainly determined by its structure matrix $A=(a_{ij}:i,j=1,2,\ldots,d)$ which means that to produce one unit of the i-th product, one requires a_{ij} units of the j-th product. Thus, once we have an input x_0, then the output x_1 in one year satisfies the equation $x_0=x_1A$. In general, we have $x_0=x_nA^n$ and then

$$x_n=x_0A^{-n}, \quad n\geq 1,$$

assuming that A is nonnegative, irreducible and invertible. This is the well-known input-output method.

Denote by $\rho(A)$ the maximal eigenvalue of A, the corresponding left- or right-eigenvectors are denoted by u (row) and v (column), respectively.

From the above simplest idealized model, one can already see the main points of Hua's optimization of global economic system (the result was appeared firstly in 1984, refer to [5,8] for a short history on the topic):

- For fastest growing rate of the system, the optimal solution of the initial input is $x_0=u$, for which we have $x_n=x_0\rho(A)^{-n}, n\geq 1$.
- If A has at least one positive diagonal element, then to keep x_n to be positive for each $n\geq 1$, the optimal solution (actually the only one) is again $x_0=u$.

The first assertion above is not so surprising, simply an application of the Perron-Frobenius theorem plus the min-max strategy. The second one is the main contribution of Hua, never appeared before as far as we know. It is even more serious that the system will be collapsed exponentially fast once $x_0\neq u$. Therefore, it is important to know the classification of the products in the economic system: the pillar products, the intermediate products and the disadvantaged products, since the system can often be collapsed at some disadvantaged products.

Following Google's PageRank (appeared in 1998), a natural way to ordering the products is using the maximal left-eigenvector u of A. However, since the matrix A in economy is quite far away from the matrix used in the network, where one has a nice graphic structure. Especially, it is far way to be a transition probability matrix. More seriously, the economic system is very sensitive, much more precise computations are required, and thus we should examine the corresponding ProductRank more carefully than the PageRank. Recall by Perron-Frobenius theorem, every nonnegative irreducible matrix A has three characteristics:

- Its maximal eigenvalue $\rho(A)$ is positive and simple.
- Its maximal left-eigenvector u is positive and one-dimensional.
- Its maximal right-eigenvector v is also positive and one-dimensional.

Note that the eigenvector u owns two of the above characteristics only, not three of them. To go further, we adopt a key transform: transforming A to a transition probability matrix P (which means that the elements of P are nonnegative and the sum of each row of P equals one).

Lemma 2.1 ([1,2,5]). *Given a positive vector w, denote by D_w the diagonal matrix with w as its diagonal elements. Next, define*

$$A_w = D_w^{-1} \frac{A}{\rho(A)} D_w.$$

Then, we have

- *A_w becomes a transition probability matrix P if and only if $w = v$.*
- *The maximal left-eigenvector of P is equal to $\mu := u \odot v$ (the vector consists of the products of the components of u and v). The normalized measure $\pi := \mu/(uv)$ is the stationary distribution of P: $\pi = \pi P^n, n \geq 1$.*

The second assertion of Lemma 2.1 shows that the left-eigenvector (or the invariant measure) μ has combined the three characteristics of A together, and hence is more essential to describe the ProductRank of A, for which we adopt μ (or equivalently π) instead of the use of u mentioned above. Moreover, $u \odot v$ owns an important economic meaning: u represents the vector of the amount of each product, v represents the vector of the true value of each product in per unit [8, Chapter 1, § 7] (often different from the price in market). Thus, $u \odot v$ gives us the vector of the total true value of each product. Hence we now have the unified unit for different products. This shows that the ProductRank here is reasonable. Furthermore, from probabilistic point of view, the stationary distribution π, as the normalized one of μ has a very important property: it is the only stationary distribution of P. For A, we do have similar stationary property that $\mu = \mu(A/\rho(A))$, but not $\mu = \mu A$ except in the unusual case that $\rho(A) = 1$. Furthermore, for P, we have the ergodic theorem

$$\lim_{n\to\infty} P^n = \mathbb{1}\pi, \tag{2.1}$$

where $\mathbb{1}$ is the column vector having constant 1 everywhere. The matrix on the right simply means that each row is the same vector π. However,

$$\lim_{n\to\infty} A^n = \lim_{n\to\infty} \rho(A)^n \left(\frac{A}{\rho(A)}\right)^n = \begin{cases} \infty, & \text{if} \quad \rho(A) > 1, \\ 0, & \text{if} \quad \rho(A) < 1, \end{cases}$$

since as an application of (2.1), it is not difficult to check that

$$\lim_{n\to\infty} \left(\frac{A}{\rho(A)}\right)^n = vu \quad \text{(a finite positive matrix).}$$

Therefore, P^n and A^n have completely different limiting behavior and hence have completely different stability. This is essential in the study on economy.

To show our ProductRank is meaningful, let us examine some practical examples. Fig. 1 is the ProductRank of 42 products produced by the input-output tables of China in 2017 (red), 2012 (blue) and 2007 (black). It covers 15 years of the economy in the country. For details, refer to the [7, Chapter 4]. We produce in our country one table in each of closed each other. Here is a remark about the input-output tables. Keeping the 2012's one at hand, the others are slightly modified for their consistence. Thus, the 24-th product is missed in 2007, which is somehow reasonable since in the earlier period, the statistical data may be missed or less completed. Hence there is a dotted black line between 23-th and 25-th products. According to the blue curve, the top 6 products are marked with blue circled numbers. It is clear that the blue and black curves have the same top 6 ranks among them. The main difference to them is the red curve, for which the top product is the 20-th one (communication, computer, etc.), but not the 12-th (chemical products). The reason is clear that the mobile phone was rapidly developed during 2012-2017. The ranks 30,33,34,35 are increasing in the three period.

The next two figures are the cumulative distribution function produced from π. We order the components $(p_k : k=1,\ldots,42)$ of π in increasing order $p_1 < p_2 < \ldots < p_{42}$. Then we obtain the discrete cumulative distribution function as $F(n): F(0)=0, F(n)=\sum_{j=1}^{n} p_j$, $F(42)=1$. From Fig. 2, one sees that the top 6 products occupy the above half of the probabilistic distribution. This is reasonable since they are the pillar products. On the other hand, from Fig. 3, one may choose the first 17 or 10 products as the disadvantaged products. These figures show the value of ProductRank for understanding the economic systems.

The Fig. 3 is a local part of Fig. 2.

We have seen the application of the transform $A \to P$ to the ProductRank. Actually, the technique was firstly used in [2] to prove the Hua's collapse theorem for economic

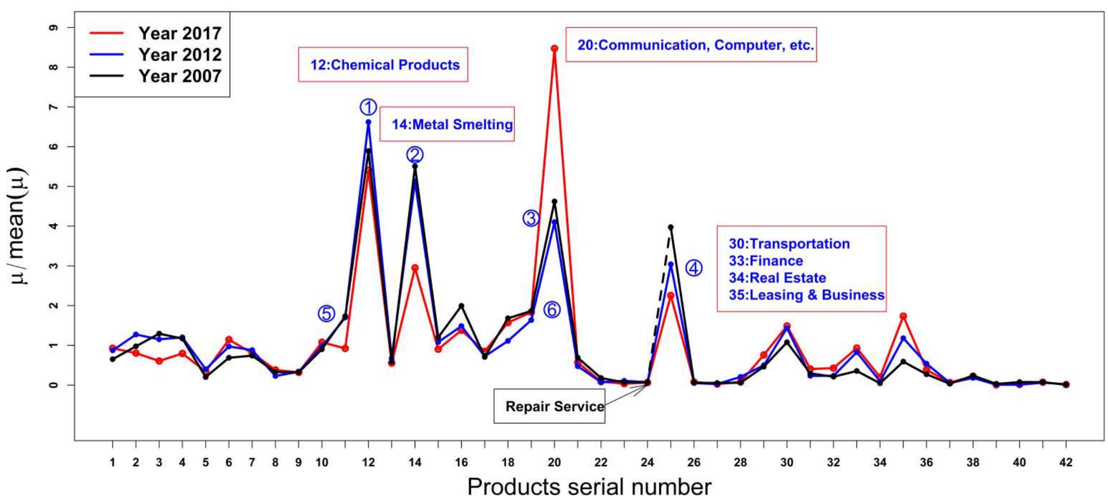

Figure 1: ProductRank by μ of 42 products in 2017, 2012, 2007.

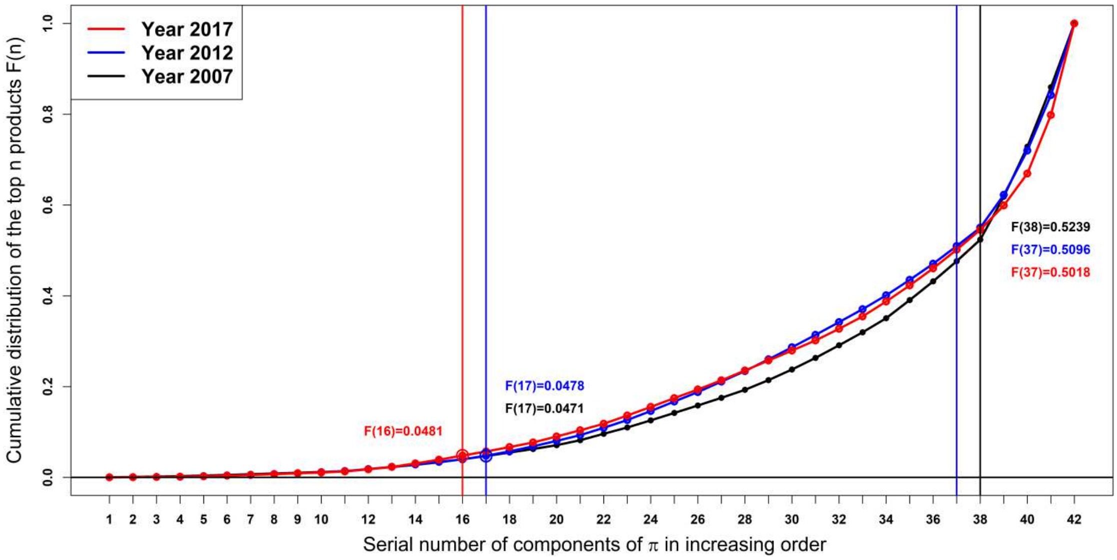

Figure 2: Cumulative distribution function of products in 2017, 2012, 2007.

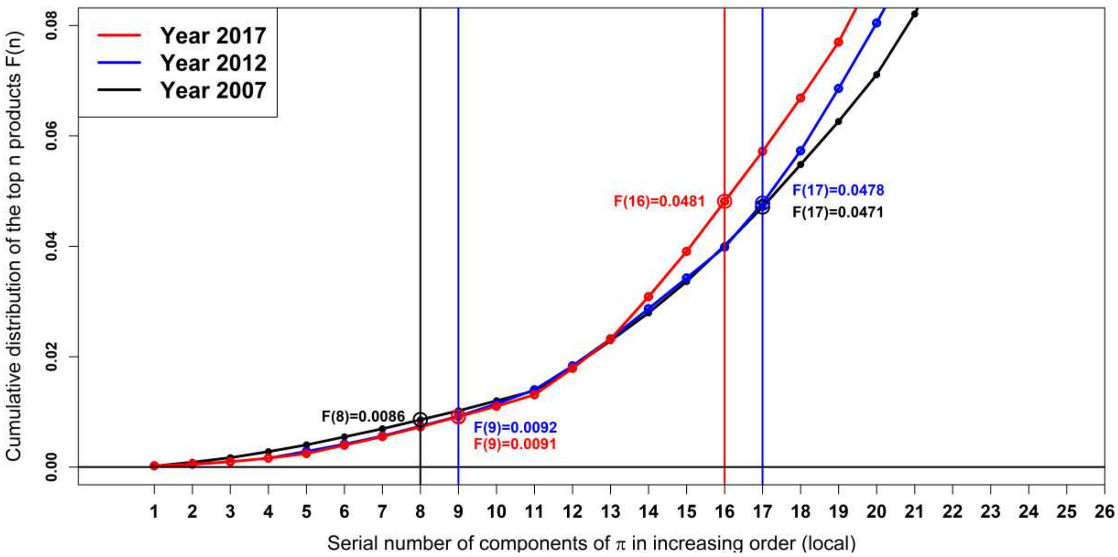

Figure 3: Cumulative distribution function of products in 2017, 2012, 2007.

system. Actually, it has much more application to the analysis on economics, including stability analysis, forecast and adjustment, algorithms of eigenpair, optimization of economic structure, and so on. Refer to [1, 5] and references therein for details. The three figures used here are taken from [10], updated partially from [1]. A new theory of the economic optimization is presented in the monograph [7].

3 Hermitian and Hermitizable matrix

We now go to complex matrix. Certainly, in such a general setup, we may have some generalized version of the Perron-Frobenius theorem, but the known results are quite restricted. However, a key point in the last section: The transform from A to P still has a meaning.

Definition 3.1. *A complex matrix A is called SR1-matrix, if $A\mathbb{1}=\mathbb{1}$. That is, the sum of each row of A equals one.*

Before moving further, we note that by using a shift if necessary, we can assume that the eigenvalue λ in the study is not zero (cf. Lemma 4.1(1) in Section 4). Thus, in what follows we assume that $\lambda \neq 0$. We may also assume if necessary that λ is simple, for instance in sorting the ProductRank.

Now, as an analog of Lemma 2.1, we have the following result.

Lemma 3.1. *Suppose that the matrix A has the right-eigenpair $(\lambda,v): Av=\lambda v$ with no zero component of v. For given vector w with no zero component, define*

$$R_w = D_w^{-1}\frac{A}{\lambda}D_w. \tag{3.1}$$

Then, we have

(1) R_w is a SR1-matrix if and only if $w=v$.

(2) R_v has the left-eigenpair $(\lambda,u\odot v):(u\odot v)R_v=\lambda(u\odot v)$.

Applying Lemma 3.1 to a Hermite matrix A, since for which, the corresponding $u=\bar{v}$ (the conjugate of v) and so $u\odot v=\bar{v}\odot v$, we obtain the following result.

Corollary 3.1. *Let A be Hermitian satisfying the hypotheses of Lemma 3.1, then the corresponding left-eigenvector of R_v equals $\bar{v}\odot v$.*

In words, the vector $\bar{v}\odot v$ combined the three characters of the Hermitian A, its eigenvalue λ and the corresponding left- and right-eigenvectors u and v. Since $\bar{v}\odot v$ represents the square of the amplitude of the wave function (equivalently, the eigenvector) v of A, we have explained the key reason why we should use "the square of the amplitude" rather than "the amplitude" only for the Born's annotation in the context of matrix mechanics. The vector $\bar{v}\odot v$ describes the ProductRank which provides not only the sort of the products but also a suitable value to each of the product, up to a factor. Equivalently, one can replace $\bar{v}\odot v$ by its normalized probability measure π and talk about the probability of a product (particle) appears, which is the same as Born's suggestion cited at the beginning of the note. Anyhow, it is just an interpretation of the same thing in two different languages, there is no objective randomness here. It is just like Einstein said, "God does not throw dice". Note that in the special case that the matrix A is nonnegative, symmetric, and $\lambda=\rho(A)$, even the "the square of the amplitude" or "the amplitude" provide the same ordering but they often have very different amplitudes. Next, since the equivalence of the matrix or wave mechanics, it follows that the same conclusion holds for the wave mechanics.

To conclude this section, we study an extension of Corollary 3.1.

Definition 3.2. *A complex matrix A is said to be Hermitizable if there is a positive measure μ such that $D_\mu A=A^H D_\mu$ (A^H is the conjugate and transpose of A). Equivalently, $\hat{A}:=D_\mu^{1/2} A D_\mu^{-1/2}$ is Hermitian.*

Refer to [3] for a criterion for the Hermitizability and for the construction of the measure μ, or refer to [6] for a short review on the topic.

Lemma 3.2. *Given a Hermitizable matrix A, we have $\hat{A}$ as shown in Definition 3.2. Corresponding to A and $\hat{A}$, define R_v and $\hat{R}_{\hat{v}}$ (where $\hat{v}$ is the right-eigenvector of $\hat{A}$), respectively, as in Lemma 3.1. Then, we have $R_v=\hat{R}_{\hat{v}}$. Hence both of them have the same left-eigenvector $\mu\odot\bar{v}\odot v$. Furthermore, the left-eigenvector of A equals $\mu\odot\bar{v}$.*

It is the position to remark that since the eigenvectors u and v of A are symmetric in the vector $u\odot v$ given in Lemmas 2.1 and 3.1, as well as in the one $\mu\odot\bar{v}\odot v$ (where $\bar{v}=u$) given in Lemma 3.2, our ProductRank is invariant under the transform: $A\to$ transpose of A.

The last result of the note is an application of the approach introduced above to the computation of eigenpairs. Actually, this is the original way in the note we come to quantum from economy. The classical approach to the goal is the power iteration (PI) and the inverse power iteration (IPI). Both are iterations of the (right-)eigenvector. However, as we have seen from Fig. 1 for instance, the eigenvector is usually very complex, oscillating. The question is: is it possible to reduce the original matrix to the one having nearly constant eigenvector? If so, then the computation should be much easier, simply start at the trivial initial vector $\mathbb{1}$. Once again, the main problem, as those discussed above, is that one does not know at the beginning a way to solve such a simple question. However, once walked up, the solution becomes quite simple: recall that we have a tool to reduce the eigenvector to be a constant $\mathbb{1}$ for a new matrix, that is Lemma 3.1. The next lemma is an extension of [5, Section 6, Lemma 16], where it is called the second quasi-symmetrizing technique.

Lemma 3.3. *As a modification of* (3.1), *define*

$$Q_w=D_w^{-1} A D_w, \tag{3.2}$$

where $v=(v^{(1)},v^{(2)},\ldots,v^{(d)})$ is the right-eigenvector of A corresponding to the eigenvalue λ and $w=(w^{(1)},w^{(2)},\ldots,w^{(d)})$ is a vector having no zero components. Then, once

$$\max_k\left|\frac{w^{(k)}}{v^{(k)}}-1\right|<\varepsilon \tag{3.3}$$

for sufficient small ε, then the right-eigenvector $w^{-1}\odot v$ of Q_w is nearly a constant vector.

Recall that the transform: either $A \to P$ or $A \to R_v$, both need to compute the right-eigenvector of A, and often require high precision. This is one of the typical applications of the above lemma. Its main function is to accelerate the convergence speed of the iterative method. This is especially important for large matrices, because the inverse power iteration used for acceleration may fail. More seriously, one may meet too large/small numbers which can not be handled by machine directly or ignored by software. Note that Q_w defined in (3.2) has only $2d^2$ pointwise products, very low computational complexity. The main steps of usage of the lemma are as follows:

- First use PI or IPI to iterate enough times or use software to compute an approximation of the eigenvector, which is recorded as w_0. In the case that w_0 is very close to a constant, then one can terminate the computation. Otherwise, go to the next step.
- Compute Q_{w_0} by (3.2). Take $\mathbb{1}$ as the initial value, and then use PI or IPI iteration for Q_{w_0} to get w_1. Again, if w_1 is very close to a constant, then one can terminate the computation.
- Repeat the above steps until the resulting vector is very close to the constant vector. Suppose we have stopped the computation at w_m, say w_3, then by Lemma 3.3, we can compute the required eigenvector v of A by the formula

$$v = w_3 \odot w_2 \odot w_1 \odot w_0. \tag{3.4}$$

For more details, refer to [5, §6] and [1, Example 9].

To conclude this section, we mention that in [3], we have proved that the spectrum of a Hermitizable matrix can be described by the spectrum of a special class of tridiagonal transition probability matrices. Once again, we have used the probabilistic language to describe the conclusion. However, there is no randomness at all, as mentioned in [3, 6]. For information along this direction, refer to the papers just cited and references therein.

4 Proofs of the results

Let us start at an elementary result.

Lemma 4.1. *Consider the right-eigenpair only.*

(1) ***Shift transformation**: The transform $\hat{A} = A + \gamma I$ (γ is a constant) makes the eigenpair (λ, g) of A to the eigenpair $(\lambda+\gamma, g)$ of $\hat{A}$. That is, the eigenvalue is changed from λ to $\lambda+\gamma$ but the eigenvector becomes the same.*

(2) ***Similar transformation**: $\hat{A} = B^{-1}AB$, where B is invertible. The eigenpair (λ, g) of A is transformed to $(\lambda, B^{-1}g)$ of $\hat{A}$. That is, the eigenvalue remains the same but the eigenvector is transformed to $B^{-1}g$.*

Proof. For the first assertion, note that

$$Ag=\lambda g \quad\Longleftrightarrow\quad (A+\gamma I)g=(\lambda+\gamma)g \quad\Longleftrightarrow\quad \hat{A}g=(\lambda+\gamma)g.$$

For the second one, note that

$$\hat{A}\hat{g}=\lambda\hat{g} \quad\Longleftrightarrow\quad B^{-1}AB\hat{g}=\lambda\hat{g} \quad\Longleftrightarrow\quad A(B\hat{g})=\lambda(B\hat{g}).$$

The proof is complete. □

Proof of Lemmas 3.1 *and* 2.1. Without loss of generality, assume that $\lambda=1$ for simplicity. Then for Lemma 3.1, we have

$$R_w\mathbb{1}=D_w^{-1}AD_w\mathbb{1}=D_w^{-1}Aw\overset{?}{=}\mathbb{1} \quad\Longleftrightarrow\quad Aw\overset{?}{=}D_w\mathbb{1}=w.$$

The last part of the line above gives us the required assertion: $w=v$. Having proved part (1) of the lemma, the part (2) follows by a simple computation.

Now, by Lemma 3.1, we obtain Lemma 2.1 immediately. □

Proof of Lemma 3.2. Applying Lemma 4.1 (2) to $B=D_\mu^{-1/2}$, it follows that for fixed eigenvalue λ, the right-eigenvector v deduces the one of $\hat{A}$: $\hat{v}=D_\mu^{1/2}v=\mu^{1/2}\odot v$. By Corollary 3.1, the corresponding left-eigenvector of $\hat{A}$ is $\hat{v}^H=\bar{v}D_\mu^{1/2}$. Hence, the first assertion of the lemma comes from

$$R_v=D_v^{-1}\frac{A}{\lambda}D_v=D_v^{-1}\frac{D_\mu^{-1/2}\hat{A}D_\mu^{1/2}}{\lambda}D_v=D_{\mu^{1/2}\odot v}^{-1}\frac{\hat{A}}{\lambda}D_{\mu^{1/2}\odot v}=D_{\hat{v}}^{-1}\frac{\hat{A}}{\lambda}D_{\hat{v}}=\hat{R}_{\hat{v}}.$$

Since $\hat{A}$ is Hermitian, by Corollary 3.1, the left-eigenvector of $\hat{R}_{\hat{v}}$ equals $\hat{v}^H\odot\hat{v}=\mu\odot\bar{v}\odot v$. This proves the second assertion of the lemma. Then the last assertion follows, which can be also verified directly:

$$(\bar{v}D_\mu)A=(\hat{v}^H\hat{A})D_\mu^{1/2}=(\lambda\hat{v}^H)D_\mu^{1/2}=\lambda(\bar{v}D_\mu).$$

The proof is complete. □

Proof of Lemma 3.3. By (3.2), we have $Q_{\mathbb{1}}=A$. Since the eigenvalue λ of A is fixed, for each w, Q_w has the same λ. In the proof below, we consider only the right-eigenvector of Q_w, denoted by g_w. Clearly, we have $g_{\mathbb{1}}=v$. Applying Lemma 4.1 (2) to $B=D_w$, it follows that the eigenvector of Q_w equals $D_w^{-1}v=w^{-1}\odot v$, which is a nearly constant vector by condition (3.2). □

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