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A Novel up to Fourth-Order Equilibria-Preserving and Energy-Stable Exponential Runge-Kutta Framework for Gradient Flows

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Abstract. In this work, we develop and analyze a family of up to fourth-order, unconditionally energy-stable, single-step schemes for solving gradient flows with global Lipschitz continuity. To address the exponential damping/growth behavior observed in Lawson's integrating factor Runge-Kutta approach, we propose a novel strategy to maintain the original system's steady state, leading to the construction of an exponential Runge-Kutta (ERK) framework. By integrating the linear stabilization technique, we provide a unified framework for examining the energy stability of the ERK method. Moreover, we show that certain specific ERK schemes achieve unconditional energy stability when a sufficiently large stabilization parameter is utilized. As a case study, using the no-slope-selection thin film growth equation, we conduct an optimal rate convergence analysis and error estimate for a particular three-stage, third-order ERK scheme coupled with Fourier pseudo-spectral discretization. This is accomplished through rigorous eigenvalue estimation and nonlinear analysis. Numerical experiments are presented to confirm the high-order accuracy and energy stability of the proposed schemes.

AMS subject classifications: 65M12, 65M15, 65M70
Key words: Gradient flows, exponential Runge-Kutta method, unconditional energy stability, error estimate.

1 Introduction

Gradient flows represent a significant class of physical models driven by free energy, characterized by a specific dissipation mechanism. Numerous challenges in fluid dynamics and material science can be effectively represented through gradient flow equations. Specifically, one main characteristic of this problem from a physical aspect is that the energy functional is decreasing all the time. Therefore, it is essential to develop ef-

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ficient and accurate numerical schemes that preserve the energy dissipation property at the discrete level. The primary objective of this paper is to develop a class of high order, unconditionally energy stable schemes for gradient flows. To illustrate the main idea, we consider a general gradient flow model with total free energy in the following form:

$$E(u)=\int_{\Omega}\frac{1}{2}u\cdot\mathcal{L}u+F(u)\mathrm{dx}, \tag{1.1}$$

where Ω is a bounded domain, $\mathcal{L}$ is a symmetric non-negative operator and $F(u)$ is a nonlinear potential function. With a specific symmetric non-negative operator $\mathcal{G}$ that commutes with $\mathcal{L}$, the gradient flow can be expressed with respect to the aforementioned energy as follows:

$$u_t=-\mathcal{G}\frac{\delta E(u)}{\delta u}=-\mathcal{G}\big(\mathcal{L}u+f(u)\big), \tag{1.2}$$

where $f(u):=F'(u)$. A common choice for the operator $\mathcal{G}$ is 1 or $-\Delta$, corresponding to the L^2 gradient flow and H^{-1} gradient flow, respectively.

Gradient flow models often exhibit strong stiffness and nonlinearity, and their steady states, which typically require long-time simulations to reach, are of considerable interest in practice. Therefore, there is a significant demand for numerical algorithms that are both efficient and accurate. Additionally, to avoid nonphysical phenomena appearing in the numerical solution, it is essential to preserve the energy dissipation during the numerical simulation. Consequently, various strategies have been designed to solve gradient flows, such as the nonlinear convex splitting scheme [8, 36, 43, 44], linear splitting (stabilization) technique [11, 34, 39], invariant energy quadratization (IEQ) method [45], and scalar auxiliary variable (SAV) algorithm [37, 38]. Among these, the nonlinear convex splitting method is known for its capacity to ensure unconditional energy stability and unique solvability. However, these methods often require solving nonlinear systems at each time step. By reformulating the energy functionals, Yang, Shen, and co-authors [37, 38, 45, 46] successively proposed the IEQ method and the SAV method. These methods can be conveniently utilized to construct schemes that are linearly solvable and unconditionally energy stable, albeit with the energy in these schemes being in a modified form. In recent years, the linear splitting (stabilization) technique has received much attention because of its capability to enlarge the stability region of linear implicit schemes. As a result, a large number of linearly implicit schemes, including the implicit-explicit Runge-Kutta (IMEX RK), exponential differencing (ETD) multi-step (MS), and ETD Runge-Kutta (ETDRK) schemes, have attracted considerable interest for solving gradient flow problems [6,21,40]. For general gradient flow equations with global Lipschitz assumptions, Fu and Yang [10] established the unconditional energy stability for a second-order stabilization ETDRK scheme with respect to the original energy. Introducing a third-order accurate Douglas-Dupont stabilization term, Cheng *et al.* [4] derived energy stability for a third-order ETDMS scheme regarding a modified energy with a few numerical correction terms. Recently, based on the global Lipschitz condition and the use of linear stabilization terms, Fu *et al.* [9] developed a four-stage, third-order stabilization