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Computational Imaging of Small-Amplitude Biperiodic Surfaces with Negative Index Material

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Abstract. This paper presents an innovative approach to computational acoustic imaging of biperiodic surfaces, exploiting the capabilities of an acoustic superlens to overcome the diffraction limit. We address the challenge of imaging physical entities in complex environments by considering the partial differential equations that govern the physics and solving the corresponding inverse problem. We focus on imaging infinite rough surfaces, specifically 2D diffraction gratings, and propose a method that leverages the transformed field expansion. We derive a reconstruction formula connecting the Fourier coefficients of the surface and the measured field, demonstrating the potential for unlimited resolution under ideal conditions. We also introduce an approximate discrepancy principle to determine the cut-off frequency for the truncated Fourier series expansion in surface profile reconstruction. Furthermore, we elucidate the resolution enhancement effect of the superlens by deriving the discrete Fourier transform of white Gaussian noise. Our numerical experiments confirm the effectiveness of the proposed method, demonstrating high subwavelength resolution even under slightly non-ideal conditions. This study extends the current understanding of superlens-based imaging and provides a robust framework for future research.

AMS subject classifications: 35R30, 74J25, 78A46

Key words: Inverse scattering problems, superlens, metamaterial, superresolution, diffraction gratings.

1 Introduction

Computational imaging is an advanced method that employs computational algorithms to extract information about physical entities from emitted or scattered waves. In contrast to direct imaging, which relies on lens systems, computational imaging offers a versatile

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approach. Widely applied in ultrasound imaging, computed tomography, and magnetic resonance imaging, it enables accurate quantitative reconstructions and access to information beyond the reach of direct imaging techniques. However, both computational and direct imaging face a common limitation – the Abbe diffraction limit [37] – which dictates that the maximum achievable resolution is approximately half the wavelength of the wave used in the imaging process.

The field of metamaterials has witnessed significant progress since Veselago [43] introduced the concept of negative-index material. Negative permittivity and permeability characterize these materials, resulting in optical properties contrary to traditional materials. Pendry [40] expanded this concept by proposing a superlens made from negative-index materials . This superlens, utilizing a material with a negative refractive index, was theorized to amplify the evanescent field, achieving unlimited resolution in theory. This concept extended to acoustic waves [28], where a negative index involves both negative density and negative bulk modulus. Theoretical propositions were supported by numerous demonstrations of negative index acoustic metamaterials [3, 14, 18, 20]. Subsequently, the idea of an acoustic superlens was proposed and validated through various methods [1, 16, 17, 21].

However, when dealing with physical entities such as obstacles or inhomogeneous mediums, as in many applications, direct imaging by a superlens becomes challenging due to complicated scattered waves. We address this challenge through computational imaging, specifically by considering the partial differential equations underlying the physics and solving the corresponding inverse problem. This paper focuses on imaging infinite rough surfaces defined by biperiodic surfaces, known as 2D diffraction gratings. These gratings find applications not only in optics but also in acoustics for manipulating sound waves [38, 42, 47].

The forward scattering problem involves determining the diffracted field given a diffraction grating and incident field. The inverse scattering problem, on the other hand, aims to reconstruct the surface profile of the grating from measured data of the diffracted field for specific incident fields. Theoretical questions regarding the inverse diffraction problems, such as uniqueness and stability, have been extensively explored [2, 5, 11, 12, 23, 26, 44]. Computational methods often involve iterative algorithms [10, 13, 22, 41, 48, 50] or direct methods based on indicator functions [4, 24, 27, 45, 46, 49].

For surfaces with small amplitude, as assumed in this paper, the inverse problem can be linearized and solved using a method based on the transformed field expansion (TFE). Originally designed for direct rough surface scattering problems [39], TFE has been extended to various contexts, providing quantitative and computationally efficient solutions [6, 7, 9, 15, 25, 29–32, 35, 36]. The small-amplitude assumption allows proving the uniqueness of the inverse problem for a single incident field, along with establishing convergence and error estimates for the computational method [8, 34].

This paper presents an imaging scheme where an acoustic superlens is placed over a biperiodic surface with small amplitude. A downward-propagating plane wave is incident upon the structure, and the field is measured on the superlens's top surface. Utiliz-