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Stochastic Runge-Kutta Methods for Preserving Maximum Bound Principle of Semilinear Parabolic Equations. Part II: Sinc Quadrature Rule

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Abstract. The maximum bound principle (MBP) is an important property for a large class of semilinear parabolic equations. To propose MBP-preserving schemes with high spatial accuracy, in the first part of this series, we developed a class of time semidiscrete stochastic Runge-Kutta (SRK) methods for semilinear parabolic equations, and constructed the first- and second-order fully discrete MBP-preserving SRK schemes. In this paper, to develop higher order fully discrete MBP-preserving SRK schemes with spectral accuracy in space, we use the Sinc quadrature rule to approximate the conditional expectations in the time semidiscrete SRK methods and propose a class of fully discrete MBP-preserving SRK schemes with up to fourth-order accuracy in time for semilinear equations. Based on the property of the Sinc quadrature rule, we theoretically prove that the proposed fully discrete SRK schemes preserve the MBP and can achieve an exponential order convergence rate in space. In addition, we reveal that the conditional expectation with respect to the Bronwian motion in the time semidiscrete SRK method is essentially equivalent to the exponential Laplacian operator under the periodic boundary condition. Ample numerical experiments are also performed to demonstrate our theoretical results and to show the exponential order convergence rate in space of the proposed schemes.

AMS subject classifications: 35B50, 60H30, 65L06, 65M12, 65M75

Key words: Semilinear parabolic equation, stochastic Runge-Kutta scheme, Sinc quadrature rule, MBP-preserving, exponential convergence rate.

1 Introduction

Consider the following initial-boundary-value problem of a semilinear parabolic partial differential equation (PDE) in the backward form:

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$$
\begin{aligned}
&u_t+\frac{1}{2}\sigma\sigma^\top:\nabla^2 u+f(u)=0, && (t,\boldsymbol{x})\in[0,T)\times D,\\
&u(t,\cdot)\text{ is }D\text{-periodic}, && t\in[0,T],\\
&u(T,\boldsymbol{x})=\varphi(\boldsymbol{x}), && \boldsymbol{x}\in\overline{D},
\end{aligned}
\tag{1.1}
$$

where $u(t,\boldsymbol{x})$ denotes the unknown function, $D=(0,a)^d\subset\mathbb{R}^d$ $(d=1,2,3)$ is a hypercube domain, f is a nonlinear operator, and the matrix $\sigma\in\mathbb{R}^{d\times d}$ is defined as

$$
\sigma=\begin{pmatrix}\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d\end{pmatrix},\quad \sigma_i\neq 0,\quad i=1,\ldots,d.
$$

It is well known that the semilinear parabolic equation (1.1) possesses the MBP in the sense that the absolute value of its solution is pointwise bounded for all time by some specific constant under appropriate initial and/or boundary conditions [9]. Up to now, great efforts have been made in developing MBP-preserving numerical methods for equations like (1.1). For the temporal discretizations, one is referred to [8–15, 18–20, 22–24, 26, 27, 33, 36, 41] and references therein. As for the spatial discretizations, a partial list of earlier works includes [2–6, 16, 17, 22, 30, 31, 37, 39, 40, 42]. Recently, the authors in [7] studied the effect of noise on the MBP-preserving property and energy evolution property of numerical methods for stochastic parabolic partial differential equation with a logarithmic Flory-Huggins potential.

Since the spectral method can not be used to construct the MBP-preserving numerical schemes for the equations like (1.1), to improve the efficiency of the long time simulations, it is important and necessary to develop some MBP-preserving schemes for (1.1) with high spatial accuracy.

In the first part of this series [35], we developed a first-order and a second-order SRK schemes with high spatial accuracy based on the probabilistic representation of the solution of (1.1) via the backward stochastic differential equation (BSDE) [29,32]. The key idea is to represent the solution of (1.1) as an integral equation via BSDE, which contains only some conditional expectations with respect to a diffusion process but not any differential operator. By applying the classical Runge-Kutta method to this integral equation, we developed a class of time semidiscrete MBP-preserving SRK methods up to fourth-order for (1.1). Since the diffusion process is a Gaussian process, one can write the conditional expectation as an integral with respect to a negative exponential function. Then we further constructed the first- and second-order fully discrete MBP-preserving SRK schemes by using the three-point Gauss-Hermite quadrature rule to approximate the integrals in the time semidiscrete schemes. Assuming that Δt is the temporal step size and $\sigma_i=\sigma_0$ for $i=1,\ldots,d$, our error analysis shows that the spatial errors of the proposed schemes in maximum norm are proportional to $\sigma_0^6\Delta t^2$ and thus can be neglected compared with the temporal errors especially for the small values of σ_0.