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A Constrained BA Algorithm for Rate-Distortion and Distortion-Rate Functions

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Abstract. The Blahut-Arimoto (BA) algorithm has played a fundamental role in the numerical computation of rate-distortion (RD) functions. This algorithm possesses a desirable monotonic convergence property by alternatively minimizing its Lagrangian with a fixed multiplier. In this paper, we propose a novel modification of the BA algorithm, wherein the multiplier is updated through a one-dimensional root-finding step using a monotonic univariate function, efficiently implemented by Newton's method in each iteration. Consequently, the modified algorithm directly computes the RD function for a given target distortion, without exploring the entire RD curve as in the original BA algorithm. Moreover, this modification presents a versatile framework, applicable to a wide range of problems, including the computation of distortion-rate (DR) functions. Theoretical analysis shows that the outputs of the modified algorithms still converge to the solutions of the RD and DR functions with rate $\mathcal{O}(1/n)$, where n is the number of iterations. Additionally, these algorithms provide ε-approximation solutions with $\mathcal{O}((MN\log N/\varepsilon)(1+\log|\log\varepsilon|))$ arithmetic operations, where M,N are the sizes of source and reproduced alphabets respectively. Numerical experiments demonstrate that the modified algorithms exhibit significant acceleration compared with the original BA algorithms and showcase commendable performance across classical source distributions such as discretized Gaussian, Laplacian and uniform sources.

AMS subject classifications: 90C25, 94A29, 94A34

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1 Introduction

The rate-distortion theory, first introduced by Shannon [26, 27], characterizes the fundamental trade-off between the rate and the distortion in lossy compression [5, 6]. Nowadays, the application scope of RD theory is extensive and far-reaching. Notably, it underpins critical technologies and standards, including image and video compression standards such as JPEG, MPEG, and H.264 standards [18, 29–31]. Additionally, the RD theory and its relevance also extend to machine learning, particularly in the realm of quantization design [4, 17] in learned lossy compression models. This multifaceted utility highlights the importance of RD theory in modern technology and research.

The computation of the RD function is of great interest and stands as a central concern within the field of RD theory. The RD function is obtained by minimizing the mutual information between the source and the reproduction subject to an average distortion constraint. Formally stated, given a source random variable $X\in\mathcal{X}$ with probability distribution P_X, a reproduction $Y\in\mathcal{Y}$ and a distortion measure $d:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}^+=[0,\infty)$, the RD function is defined as [5, 8]

$$\begin{aligned} R(D)=&\min_{P_{Y|X}} I(X;Y) \\ \text{s.t.}\quad &\mathbb{E}_{P_{XY}}[d(X,Y)]\le D. \end{aligned} \tag{1.1}$$

Here, $Y\in\mathcal{Y}$ is understood as the compressed representation, i.e. the lossy reproduction of X. The mutual information $I(X;Y)$ is minimized with respect to the conditional probability distribution $P_{Y|X}(y|x)$, and the expected distortion is constrained by a target distortion threshold D.

To date, the prevailing numerical method for computing the RD function has been the Blahut-Arimoto algorithm [3, 7], which minimizes the following RD Lagrangian:

$$\mathcal{L}^{\lambda}_{\rm RD}(P_{Y|X}):=I(X;Y)+\lambda\mathbb{E}_{P_{XY}}[d(X,Y)] \tag{1.2}$$

for each fixed multiplier $\lambda\in\mathbb{R}^+$. Geometrically, λ corresponds to the slope of the tangent line of the RD curve. The tangent point (D_λ,R_λ) associated with each fixed $\lambda\in\mathbb{R}^+$ is computed by the optimal solution $P^*_{Y|X}$ in the RD Lagrangian (1.2), i.e. $R_\lambda=I(X;Y^*)$ and $D_\lambda=\mathbb{E}[d(X,Y^*)]$ where Y^* denotes the random variable generated by $P_XP^*_{Y|X}$. Hence, keeping varying the slope, i.e. the multiplier λ, we can "sweep out" the entire RD curve.

The BA algorithm is an iterative procedure built upon this geometric view, which has been shown to converge to the RD function (e.g. [9, 10]). Various extensions (e.g. [11]) and acceleration techniques (e.g. [19, 25, 36] for channel capacity) for the BA algorithm have been studied. In addition, a mapping approach [24] has been proposed mainly for computing the RD curve for sources with continuous amplitudes. However, a common theme throughout these studies has been fixing the multiplier λ throughout iterations when minimizing the RD Lagrangian. As a consequence, to obtain the RD function $R(D)$ for a given target distortion D, these algorithms have to explore the RD curve to search for