CSIAM Trans. Appl. Math.
doi: 10.4208/csiam-am.SO-2024-0030

Vol. **6**, No. 2, pp. 380-411
June 2025

Image Denoising via Group Sparse Representations Over Local SVD and Variational Model

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Received 30 June 2024; Accepted 21 January 2025

Abstract. We propose a novel two-stage model for image denoising. With the group sparse representations over local singular value decomposition stage (locally), one can remove the noise effectively and keep the texture well. The final denoising by a first-order variational model stage (globally) can help us to remove artifacts, maintain the image contrast, suppress the staircase effect, while preserving sharp edges. The existence and uniqueness of global minimizers of the low-rank problem based on group sparse representations are analyzed and proved. Alternating direction method of multipliers is utilized to minimize the associated functional, and the convergence analysis of the proposed optimization algorithm are established. Numerical experiments are conducted to showcase the distinctive features of our method and to provide a comparison with other image denoising techniques.

AMS subject classifications: 65M32, 94A08, 65K10

Key words: Image denoising, variational model, group sparse representations.

1 Introduction

Image denoising is a classical problem in image processing. Let $\mathbf{X}$ of dimension $\sqrt{N}\times\sqrt{N}$ be an unknown clean image, $\mathbf{Y}$ be the given degraded image which is contaminated with additive white Gaussian noise (AWGN) $\mathbf{V}\in\mathbb{R}^{\sqrt{N}\times\sqrt{N}}$,

$$\mathbf{Y}=\mathbf{X}+\mathbf{V}.$$

We desire to obtain $\hat{\mathbf{X}}$ – the most likely value of $\mathbf{X}$ given the degraded image $\mathbf{Y}$. Quite a few approaches or models have been proposed to deal with this problem, such as total variation based models [29], wavelet transform approaches [27], the anisotropic diffusion

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filter [15], Perona-Malik filter [26], and nonlinear diffusion filters [20]. In this paper, we are tackling the image denoising problem from the sparse representation point of view.

This idea of learning a dictionary that yields sparse representations for a set of training image-patches has been studied in a sequence of works [1, 2, 17, 19, 30, 31]. Given a dictionary $\mathbf{D}\in\mathbb{R}^{n\times K}$ (with $K>n$, implying that it is redundant) containing prototype signal-atoms $d_k\in\mathbb{R}^n$ for $k=1,\ldots,K$, the goal of sparse representation is to represent an image patch $x\in\mathbb{R}^n$ as a linear combination of few columns (atoms) from the redundant dictionary $\mathbf{D}$, that is,

$$\hat{\alpha}=\arg\min_{\alpha}\left\{\mu\|\alpha\|_0+\|\mathbf{D}\alpha-y\|_2^2\right\},$$

where $\mathbf{x}$ and $\mathbf{y}$ are $\sqrt{n}\times\sqrt{n}$ square matrixes representing the image patch data, $\mathbf{y}=\mathbf{x}+\mathbf{v}$ is a noise version of $\mathbf{x}$, x and y are the corresponding $\mathbb{R}^n$-vector built from the columns of $\mathbf{x}$ and $\mathbf{y}$, $\alpha\in\mathbb{R}^K$ is the sparse representation vector and $\|\alpha\|_0=m\ll n$ ($\|\cdot\|_0$ is ℓ_0 norm). The denoised image is $\hat{x}=\mathbf{D}\hat{\alpha}$. Image denoising via sparse representation was first illustrated in [12]. When training the dictionary, only small dictionaries can be composed. So the key idea is to obtain a global denoising of the image by denoising overlapped local patches of the image. For an $\sqrt{N}\times\sqrt{N}$ image $\mathbf{X}$, let matrix $\mathbf{R}_{ij}$ be an $n\times N$ matrix that extracts the (i,j) block of size $\sqrt{n}\times\sqrt{n}$ from the image. In fact, there are $(\sqrt{N}-\sqrt{n}+1)^2$ such blocks. There are two ways to obtain a suitable dictionary $\mathbf{D}$, that is, either choose one of the linear transforms that sparsify the signal (e.g. Fourier, Cosine, Wavelet, etc.) or train a dictionary from a set of sample signals. Assuming $\mathbf{D}$ is the dictionary to be trained, then the maximum a posteriori (MAP) estimate will become the following form:

$$\{\hat{X},\hat{\mathbf{D}},\hat{\alpha}_{ij}\}=\arg\min_{X,\mathbf{D},\alpha_{ij}}\left\{\lambda\|Y-X\|_2^2+\sum_{ij}\mu_{ij}\|\alpha_{ij}\|_0+\sum_{ij}\|\mathbf{D}\alpha_{ij}-\mathbf{R}_{ij}X\|_2^2\right\}, \tag{1.1}$$

where X and Y are the corresponding $\mathbb{R}^N$-vector built from the columns of $\mathbf{X}$ and $\mathbf{Y}$, $\|x\|_2=(x,x)^{1/2}=(\sum_{i=1}^n x_i^2)^{1/2}$ is the 2-norm of vector x, or the Euclidean norm of vector x.

There are two stages for the iterative process, that is, a sparse coding stage followed by a dictionary update stage. An exact determination of the sparsest representations proves to be an NP-hard problem [10]. Thus, approximate solutions are considered instead, such as matching pursuit (MP) [22], the orthogonal matching pursuit (OMP) algorithms [34], basis pursuit (BP) [7], focal underdetermined system solver (FOCUSS) [28], minimax-concave penalty (MCP) [32], capped-ℓ^1 (CL1) penalty [21], smoothly clipped absolute deviation (SCAD) [14], and Log-Sum penalty (LSP) [5]. Different algorithms have been proposed for dictionary design, such as generalizing the K-means, maximum likelihood methods [17], method of optimal directions [13], maximum a posteriori probability approach [13], unions of orthonormal bases [1], K-singular value decomposition (KSVD) algorithm [1, 12], and sequential generalization of K-means [31]. One shortcoming of learning dictionaries lies in the fact that it leads to a high computational burden.

To enhance sparse representation, block matching 3-D (BM3D) has been proposed [8]. The enhancement of the sparsity is achieved by grouping similar 2D blocks of the image