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Persistence, Nonlocal Competition and Evolution of Movement: The Role of Principal Eigenvalues

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Abstract. How does the movement of individuals influence the persistence of a single species and the competition of multiple populations? Studies of such questions often involve the principal eigenvalues of the associated linear differential operators. We explore the significant roles of the principal eigenvalue by investigating two types of mathematical models for arbitrary but finite number of competing populations in spatially heterogeneous and temporally periodic environment. The interaction terms in these models are assumed to depend on the population sizes of all species in the whole habitat, representing some kind of nonlocal competition. For both models, the single species can persist if and only if the principal eigenvalue for the linearized operator is of negative sign, suggesting that the best strategy for the single species to invade when rare is to minimize the associated principal eigenvalue. For multiple populations, the global dynamics can also be completely characterized by the associated principal eigenvalues. Specifically, our results reveal that the species with the smallest principal eigenvalue among all competing populations, will gain a competitive advantage and competitively exclude other populations. This suggests that the movement strategies minimizing the corresponding principal eigenvalue are evolutionarily stable, echoing the persistence criteria for the single species.

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1 Introduction

Principal eigenvalue is a basic quantity associated with an elliptic or parabolic operator. The study of reaction diffusion equations in bounded domains often involves the principal eigenvalue of the associated linear differential operator. In this paper, we consider two types of mathematical models to illustrate the significant roles of the principal eigenvalue in determining the global dynamics of these nonlinear systems for arbitrary but finite number of competing populations. As applications, our results determine the optimal movement strategies for populations with dispersal and nonlocal competition in spatially heterogeneous and temporally varying environment.

1.1 Patch model

Consider the following single species model in K-patch landscapes:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}=d\mathbf{L}\mathbf{u}+\mathrm{diag}\{m_j(t)-u_j\}\mathbf{u}, \quad t>0, \tag{1.1}$$

where $\mathbf{u}=(u_1,\cdots,u_K)$, with $u_j(t)$ denoting the population size of the species in patch j at time t; $\mathbf{L}=(\ell_{ij})$ is a symmetric, cooperative and irreducible $K\times K$ matrix with constant entries, which satisfies $\ell_{ii}=-\sum_{j\neq i}\ell_{ij}$ for all $1\leq i\leq K$, referred as the discrete Laplacian. Here $\mathrm{diag}\{a_j\}$ denotes the $K\times K$ diagonal matrix with diagonal entries $a_1,\cdots,a_K$, and $m_j(t)$ is a T-periodic function representing the growth rate of the species in patch j, which measures the environmental heterogeneity in both space and time. Parameter $d>0$ is the migration rate of the species.

The dynamics of model (1.1) is related to the linear eigenvalue problem

$$\begin{cases}\dfrac{\mathrm{d}\boldsymbol{\varphi}}{\mathrm{d}t}=d\mathbf{L}\boldsymbol{\varphi}+\mathrm{diag}\{m_j(t)\}\boldsymbol{\varphi}+\lambda\boldsymbol{\varphi}, & t\in\mathbb{R},\\ \boldsymbol{\varphi}(t)=\boldsymbol{\varphi}(t+T), & t\in\mathbb{R},\end{cases} \tag{1.2}$$

which can be regarded as the linearization of the nonlinear model (1.1) at the trivial equilibrium $\mathbf{u}=\mathbf{0}$. By the Krein-Rutman theorem [19], problem (1.2) admits a principal eigenvalue, which is real and simple, and the corresponding eigenvector can be chosen to be positive; moreover, it has the smallest real part among all eigenvalues of (1.2).

Theorem 1.1. *For each $d>0$, let $\lambda(d)$ denote the principal eigenvalue of* (1.2)*.*

(i) If $\lambda(d)<0$, then problem (1.1) *admits a unique positive T-periodic solution*

$$\mathbf{p}=\big(p_1(t),\cdots,p_K(t)\big)>0,$$

and $u_j(t)\to p_j(t)$ as $t\to+\infty$ for all $j=1,\dots,K$.

(ii) If $\lambda(d)\geq 0$, then $u_j(t)\to 0$ as $t\to+\infty$ for all $j=1,\dots,K$.