CSIAM Trans. Life Sci.
doi: 10.4208/csiam-ls.SO-2024-0005a

Vol. **1**, No. 1, pp. 153-178
March 2025

Multiple Stable Traveling Wave Profiles of a System of Conservation Laws Arising from Chemotaxis

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Received 6 November 2024; Accepted 24 December 2024

Abstract. In this paper, we establish the existence and nonlinear stability of a hyperbolic system of conservation laws derived from a repulsive singular chemotaxis model. By the phase plane analysis alongside Poincaré-Bendixson theorem, we first prove that this hyperbolic system admits three different types of traveling wave profiles, which are explicitly illustrated with numerical simulations. Then using a unified weighted energy estimates and technique of taking anti-derivatives, we prove that all types of traveling wave profiles, including non-monotone pulsating wave profiles, are nonlinearly and asymptotically stable if the initial data are small perturbations with zero mass from the spatially shifted traveling wave profiles.

AMS subject classifications: 35C07, 35K55, 35L65, 46N60, 92C17

Key words: Chemotaxis, conservation laws, traveling waves, nonlinear stability, weighted energy estimates.

1 Introduction

Chemotaxis, the movement of an organism or entity in response to a chemical stimulus, is a widespread phenomenon in nature. One of the pioneering chemotaxis models was proposed by Keller and Segel [17] as follows to describe the wave propagation of bacterial chemotaxis:

$$\begin{cases} u_t = du_{xx} - \chi[u(\ln w)_x]_x, \\ w_t = \varepsilon w_{xx} - \mu u w^m, \end{cases} \tag{1.1}$$

where u and w denote the cell density and chemical concentration, respectively. $d>0$ and $\varepsilon \geq 0$ are cell and chemical diffusion coefficients, respectively, and $m \geq 0$ denotes

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the consumption rate. Generally chemotaxis is said to be attractive if the chemotactic coefficient $\chi>0$ and repulsive if $\chi<0$ with $|\chi|$ measuring the strength of chemotaxis. The chemotaxis model (1.1) was used in [17] to describe the process that bacteria u move up (i.e. $\chi>0$) the concentration gradient of a nutrient (denoted by w) which is absorbed by the bacteria (i.e. $\mu>0$). A prominent structural feature of (1.1) is that the chemotactic sensitivity function $\ln w$ is singular at $w=0$. This imposes tremendous challenges for both analysis and numerical computations. Though the existence/nonexistence of traveling wave solutions for any $m\geq 0$ have been well understood (cf. [29, 32]), the stability of traveling wave solutions was obtained only for the case $m=1$ and still widely remains open for the case $m\neq 1$. In the case $m=1$, if $\chi\mu>0$, the following Cole-Hopf transformation (cf. [23, 34]):

$$v=\frac{\sqrt{\chi\mu}}{\mu}\frac{w_x}{w} \tag{1.2}$$

can remove the singularity and convert the system (1.1) into a non-singular system of conservation laws as follows:

$$\begin{cases} u_t+(uv)_x=du_{xx}, \\ v_t+(u+\sigma v^2)_x=\varepsilon v_{xx} \end{cases} \tag{1.3}$$

with

$$\sigma=-\frac{\varepsilon}{\chi},$$

where we have used the rescalings $\tilde{t}=\chi\mu t$ and $\tilde{x}=\sqrt{\chi\mu}x$ to rescale the model but suppressed the tildes for simplicity. The transformed system (1.3) has no singularity and is more tractable analytically. As $\chi>0$ and hence $\sigma<0$, a large amount of interesting results have been developed to the transformed system (1.3) for both $\varepsilon>0$ and $\varepsilon=0$, such as traveling wave solutions [1–3, 19–21, 23, 24], global well-posedness of large/small solutions in the whole space [6, 14, 30, 38] or in bounded domains (or intervals) with suitable boundary conditions [22, 31, 40] and boundary layer problem [15]. When $m\neq 1$, stability results are restricted only to the spectral stability [25] and absolute instability [5] for the case $m=\varepsilon=0$ or instability [28] for $\varepsilon>0, m=0$. We also refer to a result in [8] for the existence of traveling wave solutions on a generalize Keller-Segel model. Results on the Keller-Segel model (1.1) with fractional diffusion are referred to [12, 13].

The afore-mentioned results are developed for the attractive case $\chi>0$. For the repulsive case $\chi<0$, the Keller-Segel model (1.1) with $m=1$ and $\mu<0$ was re-derived in [34] based on a random walk framework to describe the biased movement of cells that deposit signals modifying the local environment for subsequent movements, such as myxobacteria or ants that deposit non-diffusive or slowly-moving chemical substances on the way for succeeding passage. In this case, one can still use the Cole-Hopf transformation (1.2) to get (1.3) with $\sigma=-\varepsilon/\chi>0$ for which there are also many mathematical results available for the global dynamics in bounded or unbounded domain (cf. [16, 37, 41, 42]). If $\sigma\geq 0$ is allowed to be an arbitrary constant, the system (1.3) may have more applications. For