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Stability Analysis of an SIS Epidemic Model in Heterogeneous Environment

Xueli Bai[1,2], Wenping Du[3], Fang Li[4] and Maolin Zhou[5,*]

[1] *Research & Development Institute, Northwestern Polytechnical University, Shenzhen 518063, P. R. China.*
[2] *School of Mathematics and Statistics, Northwestern Polytechnical University, Xi'an 710072, P. R. China.*
[3] *Chern Institute of Mathematics, Nankai University, Tianjin 300071, P. R. China.*
[4] *School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P. R. China.*
[5] *Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, P. R. China.*

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Abstract. This paper studies an susceptible-infected-susceptible reaction-diffusion model in spatially heterogeneous environment proposed in [Allen *et al.*, Discrete Contin. Dyn. Syst., 21, 2008], where the existence and uniqueness of the endemic equilibrium are established and its stability is proposed as an open problem. However, till now, there is no progress in the stability analysis except for special cases with either equal diffusion coefficients or constant endemic equilibrium. In this paper, we demonstrate the first criterion in determining the stability of the non-constant endemic equilibrium with different diffusion coefficients. Thanks to this criterion, when one of the diffusion rates is small or large, the impact of spatial heterogeneity on the stability can be characterized based on the asymptotic behavior of the endemic equilibrium.

AMS subject classifications: Primary: 92D30, 35Q92, 35B35; Secondary: 35B40, 37N25

Key words: SIS epidemic model, endemic equilibrium, local stability, spatial heterogeneity.

1 Introduction

Partial differential equations are widely used in the modeling and analysis of the spread of infectious diseases. The impact of spatially heterogeneous environment and individual movement on the persistence or extinction of a disease has attracted a lot of studies in the

*Corresponding author. *Email address:* `zhouml123@nankai.edu.cn` (M. Zhou)

literature. We may refer to, e.g. [1–6, 10–12, 16] and the references therein. In particular, the susceptible-infected-susceptible (SIS) model is one of the most basic mathematical models for infectious disease dynamics.

Allen *et al.* [1] proposed the following frequency dependent SIS reaction-diffusion model in spatially heterogeneous environment:

$$\begin{cases} \dfrac{dS}{dt}=d_S\Delta S-\beta(x)\dfrac{SI}{S+I}+\gamma(x)I, & x\in\Omega, & t>0,\\ \dfrac{dI}{dt}=d_I\Delta I+\beta(x)\dfrac{SI}{S+I}-\gamma(x)I, & x\in\Omega, & t>0,\\ \dfrac{\partial S}{\partial \nu}=\dfrac{\partial I}{\partial \nu}=0, & x\in\partial\Omega, & t>0,\\ S(x,0)=S_0(x), \quad I(x,0)=I_0(x), & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset R^n, n\geq 1$, is a bounded domain with smooth boundary $\partial\Omega$, ν represents the unit outer normal vector on $\partial\Omega$ and $I_0(x), S_0(x)\in C(\bar{\Omega})$ are nonnegative functions satisfying $\int_\Omega I_0 dx>0$. Here $S(x,t)$ and $I(x,t)$ denote the densities of susceptible and infected individuals at location x and time t respectively, d_S and d_I are the corresponding diffusion coefficients for the susceptible and infected populations, $\beta(x)$ and $\gamma(x)$ are positive Hölder continuous on $\bar{\Omega}$ and represent the rates of disease transmission and recovery at x respectively.

Let $(\hat{S},\hat{I})$ denote, if exists, a nonnegative equilibrium solution of the problem (1.1), i.e. $(\hat{S},\hat{I})$ satisfies

$$\begin{cases} d_S\Delta S-\beta(x)\dfrac{SI}{S+I}+\gamma(x)I=0, & x\in\Omega,\\ d_I\Delta I+\beta(x)\dfrac{SI}{S+I}-\gamma(x)I=0, & x\in\Omega,\\ \dfrac{\partial S}{\partial \nu}=\dfrac{\partial I}{\partial \nu}=0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

Obviously, there are only two possibilities:

- $\hat{I}\equiv 0$ in Ω, then $(\hat{S},0)$ is called a disease-free equilibrium of the problem (1.1).
- $\hat{I}>0$ for some $x\in\Omega$, then $(\hat{S},\hat{I})$ is called an endemic equilibrium of (1.1).

The main purpose of this paper is to analyze the stability of the endemic equilibrium when it exists. The existence and uniqueness of the endemic equilibrium is thoroughly investigated in [1]. To be more specific, the basic reproduction number can be defined as follows:

$$R_0=\sup_{\varphi\in H^1(\Omega),\varphi\neq 0}\left\{\frac{\int_\Omega \beta\varphi^2 dx}{\int_\Omega d_I|\nabla\varphi|^2+\gamma\varphi^2 dx}\right\}.$$