INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 6, Pages 834–864

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Computing and Information

AN INTEGRAL EQUATION MODEL FOR PET IMAGING

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Abstract. Positron emission tomography (PET) is traditionally modeled as discrete systems. Such models may be viewed as piecewise constant approximations of the underlying continuous model for the physical processes and geometry of the PET imaging. Due to the low accuracy of piecewise constant approximations, discrete models introduce an irreducible modeling error which fundamentally limits the quality of reconstructed images. To address this bottleneck, we propose an integral equation model for the PET imaging based on the physical and geometrical considerations, which describes accurately the true coincidences. We show that the proposed integral equation model is equivalent to the existing idealized model in terms of line integrals which is accurate but not suitable for numerical approximation. The proposed model allows us to discretize it using higher accuracy approximation methods. In particular, we discretize the integral equation by using the collocation principle with piecewise linear polynomials. The discretization leads to new ill-conditioned discrete systems for the PET reconstruction, which are further regularized by a novel wavelet-based regularizer. The resulting non-smooth optimization problem is then solved by a preconditioned proximity fixed-point algorithm. Convergence of the algorithm is established for a range of parameters involved in the algorithm. The proposed integral equation model combined with the discretization, regularization, and optimization algorithm provides a new PET image reconstruction method. Numerical results reveal that the proposed model substantially outperforms the conventional discrete model in terms of the consistency to simulated projection data and reconstructed image quality. This indicates that the proposed integral equation model with appropriate discretization and regularizer can significantly reduce modeling errors and suppress noise, which leads to improved image quality and projection data estimation.

Key words. Positron emission tomography, integral equation model, image reconstruction.

1. Introduction

Positron emission tomography (PET) is a critical tool for the *in vivo* detection of cancer, due to its exquisite sensitivity to positron emitting radio-labeled molecules. In PET, patients are administered a small amounts of a radio-labeled tracer that has affinities for a particular molecular target (*e.g.*, ^{18}F-fluorodeoxyglucose (FDG) for glucose metabolism). After sufficient time for the tracer to circulate and bind (typically 1-hour for ^{18}F-FDG), the patient is then placed in a PET scanner where the photons emitted from positron annihilation are counted. These photons are emitted (nearly) back-to-back, and are counted in coincidence since their emission is correlated in time. The coincidence count data are then reconstructed via a deblurring/denoising approach based on a physical model into images that represents the tracer's biodistribution. In PET, the physical model usually considers three types of detected coincident counts, that is, true coincidences, scatter coincidences and random coincidences. True coincidences come from the same decay events that have not been scattered before hitting the detectors, whereas scatter coincidences

Received by the editors July 5, 2021.
2000 *Mathematics Subject Classification.* 65R20, 65R32.
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are the events that have been scattered. Random coincidences involve two separate decay events occurring within a physically allowable timing window (coincidence window, typically 2-5 ns). In current practice, a reduced model, based on the true coincidences portion of the physical model, is generally used and discussed in the literature. In this reduced model, the scatter and random counts are externally estimated as additive count corrections. Discretization based on a piecewise constant approximation of the reduced model leads to the commonly used discrete system model. However, the resulting discrete model introduces irreducible modeling error due to its use of the piecewise constant approximation. Moreover, even though a great deal of work has been done in investigating the reduced model, the explicit development of higher accuracy representations of the physical model and its subsequent regularization is less explored.

As mentioned in a comprehensive review [10], substantial focus has been placed on improving the accuracy of the physical models. Accurately modeling the physical effects can improve the fidelity of the model, and further increase the quality (such as spatial resolution, lesion contrast) of the reconstructed image. If the discrete nature of the projection data is ignored, the attenuated Radon transform model (ART) [7, 33] represents an idealized and accurate model for true coincidences. However, in a real PET data acquisition, a new form of the ART model that accounts for the discreteness of data is required. Existing models for true coincidences, mainly factorization approaches, can be considered as the discretization of the ART model and still have the non-negligible modeling errors. Factorization approaches [4, 24, 26, 32] formulate the system matrix as the product of independent sparse matrices (*e.g.*, attenuation, positron range and detector efficiency). This leads to a sparse system matrix, which substantially reduces storage space and can help reduce reconstruction times. However, these factorizations are based on the decoupling of the integral kernel that can introduce modeling errors that fundamentally limit image quality. These models also fail to consider the continuous nature of the problem and can be regarded as piecewise constant approximations of the integral equation model. The data/model mismatch mentioned above results in irreducible modeling error and thus imposes a fundamental bottleneck in the improvement of image quality. Therefore, it is the goal of this study to reduce the modeling error through a holistic approach via establishing a continuous physical model and its discretization of higher order accuracy.

In this study, we propose an integral equation model combined with the discretization, regularization, and optimization algorithm, leading to a new PET image reconstruction method. In contrast to existing discrete models (DM), the true coincidences model proposed in this paper improves consistency with the underlying physics and geometry of the PET imaging. We consider the attenuation and effective detection angle of each point source to determine the contribution weights of the true coincidences. The model is then discretized by using piecewise polynomials, resulting in a novel discrete system for PET reconstruction. Piecewise polynomials can more precisely express the tracer distribution than a piecewise constant function, and result in a higher order of accuracy than a piecewise constant function. Based on the smoothness of the tracer distribution function, we designed a suitable sparse wavelet-based regularizer to penalize the approximation error of the discretization and to treat the ill-posedness of the proposed piecewise