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A DIFFERENCE FINITE ELEMENT METHOD FOR CONVECTION-DIFFUSION EQUATIONS IN CYLINDRICAL DOMAINS

CHENHONG SHI, YINNIAN HE, DONGWOO SHEEN*, AND XINLONG FENG

Abstract. In this paper, we consider 3D steady convection-diffusion equations in cylindrical domains. Instead of applying the finite difference methods (FDM) or the finite element methods (FEM), we propose a difference finite element method (DFEM) that can maximize good applicability and efficiency of both FDM and FEM. The essence of this method lies in employing the centered difference discretization in the z-direction and the finite element discretization based on the P_1 conforming elements in the (x, y) plane. This allows us to solve partial differential equations on complex cylindrical domains at lower computational costs compared to applying the 3D finite element method. We derive stability estimates for the difference finite element solution and establish the explicit dependence of H_1 error bounds on the diffusivity, convection field modulus, and mesh size. Finally, we provide numerical examples to verify the theoretical predictions and showcase the accuracy of the considered method.

Key words. Convection-diffusion equation, difference finite element method, cylindrical domain, error estimates.

1. Introduction

In this paper, we consider the difference finite element method (DFEM) to the following convection-diffusion equation with the homogeneous Dirichlet boundary condition:

$$-\alpha\widehat{\Delta}u(\mathbf{x},z) + \widehat{\boldsymbol{\beta}}\cdot\widehat{\boldsymbol{\nabla}}\,u(\mathbf{x},z) = f(\mathbf{x},z), \quad (\mathbf{x},z)\in\Omega, \tag{1a}$$

$$u(\mathbf{x},z) = 0, \quad (\mathbf{x},z)\in\partial\Omega. \tag{1b}$$

Here, and in what follows, frequently we use the notation $\mathbf{x} = (x, y)$. The unknown is a function $u : \overline{\Omega} \to \mathbb{R}$, $\overline{\Omega}$ is the closure of the open set $\Omega = \omega \times [a_3, b_3]$, $\alpha > 0$ is the constant diffusivity, $\widehat{\boldsymbol{\beta}} = (\boldsymbol{\beta}, \beta_3) = (\beta_1, \beta_2, \beta_3)$ is the given convection field satisfying that the components are constants and the RHS function $f : \Omega \to \mathbb{R}$ is the given source function. In a quest for greater clarity, we use the following notation $\widehat{\Delta} = \partial_{xx} + \partial_{yy} + \partial_{zz}$ and $\widehat{\boldsymbol{\nabla}} = (\partial_x, \partial_y, \partial_z)^\top$.

The finite element method (FEM) and the finite difference method (FDM) are two traditional important methods to solve partial differential equations (PDEs) using computers. FEMs are more adequate to handle PDEs with irregular coefficients and boundary conditions prescribed on complex geometric shapes, and thus can be used for modeling complex physical problems, but more expensive computation costs are needed especially for high-dimensional problems. On the other hand, FDMs have clear advantages in their implementation and low computing cost, but FDMs that require high regularity of solutions to the governing PDEs have certain

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*Corresponding author.

limitation in direct application. For references, the reader may suggested to consult [2, 4, 9, 22, 21, 19, 3, 6, 1, 7, 15, 18], and the references therein.

Based on these, it is natural to combine these two methods to maximize applicability and efficiency to solve a certain suitable class of problems that bear both benefits of FEM and FDMs. Such cases occur, for instance, in dealing most problems with cylidrical domains whose underlying base geometries are complicate. In this spirit, the idea of difference finite element methods (DFEM) have been developed in recent years [14].

In [14], the authors proposed the Difference Finite Element Method (DFEM) for solving the 3D Poisson equation. The method utilizes a combination of the finite difference discretization in the z-direction and the finite element discretization in the (x, y)-domain ω using the P_1-conforming elements. In DFEM, the numerical solution of the 3D Poisson equation is obtained by solving a series of $2D$ elliptic equations, thereby reducing the computational complexity. Specifically, the coefficient matrix only needs to be computed in a $2D$ domain ω, making the overall computation more efficient. In this paper, our work are to discretize the convection-diffusion equation in a 3D domain using the Difference Finite Element Method (DFEM) and explicitly provide the matrix representation of the DFE discretization of the gradient term. This allows us to use the finite element method in the (x, y) plane where high flexibility and strong adaptability are required, and use the finite difference method in the z-direction to save computation cost and reduce implementation difficulty. Superconvergence in H_1 norm of this approach was studied in [10]. Since then, the idea of DFEM has been applied to solve 3D steady state Stokes and Navier-Stokes problems [17, 16, 11, 12].

We are interested in further development of DFEM for the convection-diffusion equation particularly in cylindrical domains. FDM is applied in the lateral direction while FEM is applied in the longitudinal 2D domain.

The remaining part of this paper is structured as follows. In Section 2, we recall the FE methods and establish the stability and error estimates for the 2D steady convection-diffusion problems. In Section 3, we present the DFE method based on the P_1-element for the z-direction discretization of the 3D steady convection-diffusion problems and perform stability and error estimates. In Section 4, we define the DFE solution pair $u_{h\tau}$ based on the $P_1 \times P_1$-element of the 3D steady convection-diffusion equation and prove the first order H_1-error bound of the DFE solution pair $u_{h\tau}$ with respect to the solution u of the 3D steady convection-diffusion equation. In Section 5, several numerical examples are presented to illustrate the effectiveness of the proposed method. Finally, the conclusions are drawn in Section 6.

1.1. Notations. For measurable set S in $\mathbb{R}^d$, by $(\cdot,\cdot)_S$ we denote the $L^2(S)$ inner product. For $k \in \mathbb{Z}$, standard notations for Sobolev spaces $H^k(S)$ will be employed. By $\|v\|_{k,S}$ and $|v|_{k,S}$ we mean the standard Sobolev norms and seminorms for $H^k(S)$. $\langle\cdot,\cdot\rangle_{X',X}$ will mean the duality paring between the topological vector space X and its dual X'. However, wherever there is no confusion, the subscripts may be omitted.