INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 4, Pages 459–475

doi: 10.4208/ijnam2024-1018

A STABILIZER-FREE WEAK GALERKIN FINITE ELEMENT METHOD FOR THE DARCY-STOKES EQUATIONS

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Abstract. In this paper, we propose a new method for the Darcy-Stokes equations based on the stabilizer-free weak Galerkin finite element method. In the proposed method, we remove the stabilizer term by increasing the degree of polynomial approximation space of the weak gradient operator. Compared with the classical weak Galerkin finite element method, it will not increase the size of global stiffness matrix. We show that the new algorithm not only has a simpler formula, but also reduces the computational complexity. Optimal order error estimates are established for the corresponding numerical approximation in various norms. Finally, we numerically illustrate the accuracy and convergence of this method.

Key words. Stabilizer-free, weak Galerkin finite element method, Darcy-Stokes equations, weak gradient operator, optimal order error estimates.

1. Introduction

Darcy-Stokes equations has widely applications in groundwater protection, oil extraction, geophysics and etc. The research on the numerical solution of Darcy-Stokes equations has been attracting much more attention in recent years. The mixed finite element method[5, 13] is a common numerical method for solving fluid problems. However, the mixed finite element method must satisfy the inf-sup condition[1], which restricts the choice of finite element space pairs. For example, a finite element space pair of equal order will not satisfy this condition. The stabilized finite element methods have been proposed to solve the above problem. These stabilized methods are constructed by adding stabilizer terms to the classical mixed finite element formulations, for example, a least squares terms, which obtained by bubble condensation contain coefficients related to the mesh size and equation parameters. On the other hand, untraditional finite element methods are developed for solving PDEs, which used discontinuous functions as approximation functions. The Discontinuous Galerkin finite element method (DG) was first proposed by Reed and Hill in 1973. Based on this method, various DG methods have been proposed: the Local Discontinuous Galerkin finite element method[2, 4], the Hybrid Discontinuous Galerkin finite element method[7, 30], the selective immersed discontinuous Galerkin method[16], etc. These above methods have some good features like local conservation of physical quantities and flexibility in meshes.

The weak Galerkin(WG) finite element method was first proposed by Wang and Ye for second-order elliptic problems. Subsequently, they have been used for solving Stokes[19, 20, 32], Darcy-Stokes[21, 28], Brinkman[14, 29], convection-diffusion-reaction equations[25] and so on. It is well known that the WG finite element method has two main features. One is the use of discontinuous piecewise polynomials in the finite element space. Therefore, this method is suitable for general polygonal or polyhedral meshes, and it is easy to construct finite element spaces. The other feature is that the differential operators are approximated by weak forms

Received by the editors on August 11, 2023 and, accepted on February 25, 2024.
2000 *Mathematics Subject Classification.* 65N12, 65N30.
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which are locally-defined on each element. For example, gradient operator ∇ and divergence operator $\nabla\cdot$ are approximated by weak gradient operator ∇_w and weak divergence operator $\nabla_w\cdot$ respectively. Thus, it is straightforward for building the discretization scheme. Due to these advantages, the WG finite element method has been widely developed, various WG finite element methods have been proposed, such as modified WG finite element method [18, 26], hybridized WG finite element method [22, 39], and so on. It is worth to mention, all the above methods contain a stabilizer which enforces a certain weak continuity in their numerical schemes. Recently, a new WG finite element method, called the stabilizer-free weak Galerkin (SFWG) finite element method, was derived for second elliptic equations on polytopal mesh[33]. The main idea of this SFWG method is raise the degree of polynomials used to compute weak gradient ∇_w. Removing stabilizers from the WG finite element method simplifies fomulations and reduces progrmming complexity. Based on this idea, in [36, 37], a new SFWG method was employed for the Stokes equations. Especially, the method in [37] not only achieved exact divergence free velocity field, but also obtained pressure-robustness.

Now, let us turn to the Darcy-Stokes model which is established for free fluids and coexisting flow system. In this paper, we consider the following problem: seek unknown functions $\boldsymbol{u}$ and p satisfying

$$\begin{cases} -\epsilon^2\Delta\boldsymbol{u} + \boldsymbol{u} + \nabla p = \boldsymbol{f} & in\ \Omega, \\ \nabla\cdot\boldsymbol{u} = 0 & in\ \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & on\ \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a polygonal ($d = 2$) or a polyhedral domain ($d = 3$). Here ϵ is the fluid viscosity coefficient, $\boldsymbol{f}$ and $\boldsymbol{g}$ are some given data. Throughout this paper, we assume that $\boldsymbol{g} = \boldsymbol{0}$.

The weak formulation for the Darcy-Stokes equations (1) is finding $\boldsymbol{u} \in [H_0^1(\Omega)]^d$ and $p \in L_0^2(\Omega)$ that satisfying

$$\epsilon^2(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + (\boldsymbol{u}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \tag{2}$$

$$(\nabla\cdot\boldsymbol{u}, q) = 0, \tag{3}$$

for all $\boldsymbol{v} \in [H_0^1(\Omega)]^d$ and $q \in L_0^2(\Omega)$.

In [28], a WG finite element method was proposed for the Darcy-Stokes equations (1). They proved the inf-sup condition and derived the optimal order error estimates. However, the new scheme contained a stabilizer. The goal of this paper is to continue the investigation of SFWG finite element methods for the Darcy-Stokes equations. By using high order polynomials to compute the weak gradient operator ∇_w and weak divergence operator $\nabla_w\cdot$, we introduce a new WG finite element method without stabilizers. Although raising the degree of polynomials in the computation of weak gradient operator and weak divergence operator, it will not increase the size of global stiffness matrix. In addition, the proposed method will reduce the computational complexity of programming and achieve the optimal order of convergence.

This paper is organised as follows. In the next section, it presents a SFWG finite element method for the Darcy-Stokes equations (1). In Section 3, we give some essential lemmas and prove the existence and uniqueness of the solution of the new method. In Section 4, some error equations are derived. In Section 5, we derive the error estimates of the $|||\cdot|||$ norm and L^2 norm of the velocity function and the L^2 norm of the pressure function. The sixth Section shows some numerical experiments.