INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 4, Pages 476–503

doi: 10.4208/ijnam2024-1019

A VOIGT-REGULARIZATION OF THE THERMALLY COUPLED INVISCID, RESISTIVE MAGNETOHYDRODYNAMIC

XINGWEI YANG, PENGZHAN HUANG*, AND YINNIAN HE

Abstract. In this paper, we prove the existence of weak solution and the uniqueness of strong solution to a Voigt-regularization of the three-dimensional thermally coupled inviscid, resistive MHD equations. We also propose a fully discrete scheme for the considered problem, which is proven to be stable and convergent. All computational results support the theoretical analysis and demonstrate the effectiveness of the presented scheme.

Key words. Thermally coupled magnetohydrodynamic, inviscid, resistive, Voigt-regularization, finite element method, three-dimensional MHD equations.

1. Introduction

The incompressible magnetohydrodynamic (MHD) describes the dynamic behavior of an electrically conducting fluid under the influence of a magnetic field, and has a wide range of applications in scientific and engineering, such as electromagnetic pumping, liquid metal, electrolyte, and so on (see [1, 2, 3, 4, 5]). It consists of a viscous, incompressible fluid which owns the property of electric current conduction and interacting with electromagnetic induction. The MHD flow has a multi-physics phenomenon: the magnetic field changes the momentum of the fluid through the Lorenz force, and conversely, the conducting fluid influences the magnetic field through electric currents. Additionally, if the buoyancy effect cannot be neglected in the momentum equation due to temperature differences in the conductive flow, then the incompressible MHD equations are usually coupled to the heat equation. In this way, multiple physical fields (velocity, pressure, magnetic and temperature) will be coupled in the MHD system.

Usually, the thermally coupled incompressible MHD system is given as follows [6]:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla(p + \frac{1}{2}|\boldsymbol{B}|^2) - (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} = \boldsymbol{f} + \boldsymbol{\beta}\theta, \tag{1a}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{1b}$$

$$\boldsymbol{B}_t - \mu\Delta\boldsymbol{B} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} + \nabla q = \nabla\times\boldsymbol{g}, \tag{1c}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{1d}$$

$$\theta_t - \kappa\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = \Psi, \tag{1e}$$

with appropriate boundary and initial conditions. Here, $\nu \geq 0$ is the fluid viscosity, $\mu \geq 0$ is the magnetic resistivity, κ is the thermal conductivity, $\boldsymbol{\beta}$ is the thermal expansion coefficient, and the unknowns are the fluid velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, the fluid pressure $p(\boldsymbol{x}, t)$, the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$, the magnetic pressure $q(\boldsymbol{x}, t)$, and the temperature field $\theta(\boldsymbol{x}, t)$. In fact, by *a posteriori*, one can drive that $\nabla q \equiv \mathbf{0}$. Besides, the given function $\boldsymbol{f}$ is the external force, $\boldsymbol{g}$ is the known applied current, and Ψ is the heat source. Note that these equations contain the three dimensional

Received by the editors on December 14, 2023 and, accepted on April 16, 2024.
2000 *Mathematics Subject Classification.* 65M60, 76W05.
*Corresponding author.

Navier-Stokes equations as a special case (namely $\boldsymbol{B} \equiv \mathbf{0}$ and $\theta \equiv 0$), and the mathematical theory is far from complete.

Denote by $P := p + \frac{1}{2}|\boldsymbol{B}|^2$ a modified pressure, in this paper we study the following Voigt-regularization of (1) in inviscid and resistive case (i.e., $\nu = 0$ and $\mu \neq 0$).

$$\boldsymbol{u}_t - \alpha^2 \Delta \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla P - (\boldsymbol{B} \cdot \nabla)\boldsymbol{B} = \boldsymbol{\beta}\theta, \tag{2a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2b}$$

$$\boldsymbol{B}_t - \mu \Delta \boldsymbol{B} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} - (\boldsymbol{B} \cdot \nabla)\boldsymbol{u} + \nabla q = 0, \tag{2c}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2d}$$

$$\theta_t - \kappa \Delta \theta + \boldsymbol{u} \cdot \nabla \theta = 0, \tag{2e}$$

$$(\boldsymbol{u}, \boldsymbol{B}, \theta)|_{t=0} = (\boldsymbol{u}_0, \boldsymbol{B}_0, \theta_0), \tag{2f}$$

where $\alpha^2 \Delta \boldsymbol{u}_t$ is the Voigt term and $\alpha > 0$ is a regularization parameter. Moreover, when $\alpha = 0$, we formally retrieve (1) by adding forcing terms to (2a), (2c) and (2e), and reintroducing a viscous term $\nu \Delta \boldsymbol{u}$ to the right-hand side of (2a).

The Voigt (also written Voight) term was originally proposed by Voigt in [7] for viscoelastic fluids. Viscoelasticity is the property of a material that, under stress and deformation, exhibits both viscous and elastic characteristics. Unlike the Kelvin stress-strain relation [8], Voigt has derived a system of equations that governs the behavior of elastic solids with viscous properties, which is known today as the Kelvin-Voigt equations [7]. In [9], the Navier-Stokes-Voigt (NSV) equations were firstly introduced by Oskolkov as a model of Kelvin-Voigt fluids in which α denotes a material parameter connected to a characteristic length of viscoelastic effects. The authors also pointed out that the NSV equations have a real physical sense, and describe the flow of a viscous incompressible Newtonian fluid which requires $\frac{\alpha^2}{\nu}$ units of time in order to be set in motion under the action of a suddenly applied force. The NSV equations were proposed by Cao et al. [10] as a regularizaion of the Euler equations and Navier-Stokes equations which suggested a smaller resolution requirement in large scale computations.

In the last several decades, many analyses and applications regarding the Voigt regularization have been studied (see, e.g., [11, 12, 13, 14, 15, 16, 17]). The Voigt regularization enjoys a feature that it is inviscid and does not require any artificial boundary conditions to prove the global existence and uniqueness of strong solutions [10]. It is also simpler than the nonlinear viscosity model of Ladyzhenskaya [18] and Smagorinsky [19]. Due to these benefits, the ability to adapt an existing CFD code to the Voigt regularization without intrusion has great interest. In [20], Kuberry et al. proposed a Voigt regularization algorithm for the Navier-Stokes equations. Numerical tests show that the Voigt regularization algorithm on a coarse mesh produces good approximations to higher Reynolds number. Later on, Layton and Rebholz [21] found that the regularization parameter α has effect of slowing the temporal evolution; that is, compared to the usual solutions of the Navier-Stokes equations, the NSV approximations have a longer relaxation time and damped effects decay more slowly. The statistical properties of the NSV model have also been investigated computationally, using a phenomenological model of turbulence known as the Sabra shell model [22]. The results indicate that the NSV model may capture important statistical features of the Navier-Stokes equations, and therefore give motivation for it to be investigated for use in numerical simulations. Furthermore, in the context of numerical computations, the NSV system appears to have