INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 21, Number 4, Pages 528–559

doi: 10.4208/ijnam2024-1021

hp-VERSION ANALYSIS FOR ARBITRARILY SHAPED ELEMENTS ON THE BOUNDARY DISCONTINUOUS GALERKIN METHOD FOR STOKES SYSTEMS

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Abstract. In the present work, we examine and analyze an *hp*-version interior penalty discontinuous Galerkin finite element method for the numerical approximation of a steady fluid system on computational meshes consisting of polytopic elements on the boundary. This approach is based on the discontinuous Galerkin method, enriched by arbitrarily shaped elements techniques as has been introduced in [13]. In this framework, and employing extensions of trace, Markov-type, and H^1/L^2-type inverse estimates to arbitrary element shapes, we examine a stationary Stokes fluid system enabling the proof of the inf/sup condition and the *hp*- a priori error estimates, while we investigate the optimal convergence rates numerically. This approach recovers and integrates the flexibility and superiority of the discontinuous Galerkin methods for fluids whenever geometrical deformations are taking place by degenerating the edges, facets, of the polytopic elements only on the boundary, combined with the efficiency of the *hp*-version techniques based on arbitrarily shaped elements without requiring any mapping from a given reference frame.

Key words. Arbitrarily shaped elements, discontinuous Galerkin finite element method, hp-version stability, a-priori error estimates, fluid dynamics.

1. Introduction

Recent years have shown scientists' interest significantly focused on the context of Galerkin finite element methods. This effort has given birth to new methods based on general-shaped elements which arise computational complexity reduction, like mimetic finite difference methods [9], virtual element methods [8], various discontinuous Galerkin approaches such as interior penalty Galerkin methods [14], and hybridized discontinuous Galerkin [15, 18], which are very attractive and used by the engineering and mathematics community. We continue by reporting more works related to discontinuous Galerkin (dG) methods, similar finite element frameworks, and advances, *h*- or *hp*- version, fluid and Stokes systems related literature. We also introduce the skeleton of the present work. Hence, other approaches have involved non-polynomial approximation spaces like polygonal and other generalized finite element methods, [26, 54]. We refer to [14] for admissible polygonal/polyhedral element shapes for which the general interior penalty discontinuous Galerkin method (IP-dG), appears both stable and convergent while generalizes under mild assumptions the validity of standard approximation results, such as inverse estimates, best approximation estimates, and extension theorems.

In a *p*-version Galerkin framework achieving exponential convergence, for smooth partial differential equation problems defined on generally curved domains using isoparametrically mapped elements, we recall [46, 47], while for non-linear maps on element patches that are used to represent domain geometry we refer to [45, 53]. Although in both cases, as the polynomial order increases, the aforementioned mapping appears very costly and/or difficult to construct and implement in practice.

Received by the editors on September 7, 2023 and, accepted on April 30, 2024.
2000 *Mathematics Subject Classification.* 35Q35, 76M10, 35Q30.

For Stokes flow systems, in [4, 55], an *hp*-discontinuous Galerkin approximation that shows better stability properties than the corresponding conforming ones is examined with finite element triangulation not required to be conforming employing discontinuous pressures and velocities, while it is defined on the interfaces between the elements involving the jumps of the velocity and the average of the pressure. The work of [28] also, describes a family of dG finite element methods formulated and analyzed for Stokes and Navier-Stokes problems introducing the good behavior of the inf-sup and optimal energy estimates for the velocity and pressure. In addition, this method can treat a finite number of non-overlapping domains with non-matching grids at interfaces. In [16, 32, 52], Stokes system local discontinuous Galerkin methods for a class of shape regular meshes with hanging nodes is investigated, as well as, several mixed discontinuous Galerkin approximations with their a priori error estimates. In [6], a discontinuous Galerkin (dG) approach to simulations on complex-shaped domains, using trial and test functions defined on a structured grid with essential boundary conditions imposed weakly, where the discretization allows the number of unknowns to be independent of the complexity of the domain. [44] concerns an unfitted dG method proposing to discretize elliptic interface problems, where *h*- and *hp*- error estimates and convergence rates are proved. The authors of [56], treat an unfitted dG method for the elliptic interface problems, based on a variant of the local dG method, obtaining the optimal convergence for the exact solution in the energy norm and its flux in the L^2 norm. In [7] an unfitted discontinuous Galerkin method for transport processes in complex domains in porous media problems is examined, allowing finite element meshes that are significantly coarser than those required by standard conforming finite element approaches. Further, in [25] an advection problem is developed based on an unfitted discontinuous Galerkin approach where the surface is not explicitly tracked by the mesh which means the method is flexible with respect to geometry efficiently capturing the advection driven by the evolution of the surface without the need for a space-time formulation, back-tracking trajectories or streamline diffusion. Finally, in [24] a linear transport equation on a cut cell mesh using the upwind discontinuous Galerkin method with piecewise linear polynomials and with a method of lines approach is presented employing explicit time-stepping schemes, regardless of the presence of cut cells.

In addition, various classes of fitted and unfitted mesh methods for interface or transmission problems may be seen as generalized concepts of mesh elements, as well as, several unfitted finite element methods have been proposed in recent years, indicatively we mention the unfitted boundary finite element methods [5] and immersed finite element methods [41]. More extensively, an optimally convergent method of fictitious type domains avoiding the numerical integration on cut mesh elements for a Poisson system has been introduced in [40], while in [29] a method for the finite element solution of the elliptic interface problem, using an approach due to Nitsche is proposed allowing discontinuities internal to the elements approximating the solution across the interface. In addition, from a reduced basis for unfitted mesh methods point of view, evaluating the fixed background mesh used in immersed and unfitted mesh methods, parametrized Stokes and other flow systems have managed to be solved using a unified reduced basis presenting the flexibility of such methods in geometrically parametrized Stokes, Navier-Stokes, Cahn-Hilliard systems as in [33–37]. For an immersed interface method for discrete surface representations employing accurate jump conditions evaluated along interface representations using projections, one could see [38], and for a ghost fluid method coupled with a volume