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A HIGH ORDER UNFITTED FINITE ELEMENT METHOD FOR TIME-HARMONIC MAXWELL INTERFACE PROBLEMS

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Abstract. We propose a high order unfitted finite element method for solving time-harmonic Maxwell interface problems. The unfitted finite element method is based on a mixed formulation in the discontinuous Galerkin framework on a Cartesian mesh with possible hanging nodes. The H^2 regularity of the solution to Maxwell interface problems with C^2 interfaces in each subdomain is proved. Practical interface-resolving mesh conditions are introduced under which the hp inverse estimates on three-dimensional curved domains are proved. Stability and hp a priori error estimate of the unfitted finite element method are proved. Numerical results are included to illustrate the performance of the method.

Key words. Maxwell interface problem, high order unfitted finite element method, hp a priori error estimate.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary Σ. We consider in this paper the following time-harmonic Maxwell interface problem

$$\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}) - k^2\varepsilon \boldsymbol{E} = \boldsymbol{J}, \quad \mathrm{div}\,(\varepsilon \boldsymbol{E}) = 0 \quad \text{in } \Omega, \tag{1}$$

$$[\![\boldsymbol{E} \times \boldsymbol{n}]\!]_\Gamma = 0, \quad [\![(\mu^{-1}\nabla \times \boldsymbol{E}) \times \boldsymbol{n}]\!]_\Gamma = 0, \quad [\![\varepsilon \boldsymbol{E} \cdot \boldsymbol{n}]\!]_\Gamma = 0 \quad \text{on } \Gamma, \tag{2}$$

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{on } \Sigma, \tag{3}$$

where $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ with $\mathrm{div}\,\boldsymbol{J} = 0$ in Ω and $\boldsymbol{g} \times \boldsymbol{n} \in \boldsymbol{H}^{3/2}(\Sigma)$. Here and throughout the paper, for any Banach space X, we denote $\boldsymbol{X} = X^3$ and $\|\cdot\|_X$ both the norms of X and $\boldsymbol{X}$.

We assume the domain Ω is divided by a C^2 interface Γ into two subdomains so that $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ and Ω_1 is strictly included in Ω. For simplicity, we assume the relative permeability μ and the relative permittivity ε are piecewise constants $\mu = \mu_1\chi_{\Omega_1} + \mu_2\chi_{\Omega_2}$, $\varepsilon = \varepsilon_1\chi_{\Omega_1} + \varepsilon_2\chi_{\Omega_2}$, where for $i = 1, 2$, χ_{Ω_i} is the characteristic function of Ω_i, and μ_i, ε_i are positive constants. $k = \omega\sqrt{\varepsilon_0\mu_0}$ is the wave number of the vacuum with $\omega > 0$ the angular frequency and μ_0, ε_0 the permeability and permittivity of the vacuum. With this notation, $\boldsymbol{J} = \mathbf{i}k\mu_0\boldsymbol{J}_a$ with $\boldsymbol{J}_a$ being the applied current density. We denote by $\boldsymbol{n}$ both the unit outer normal to Ω_1 on Γ and the unit outer normal to Ω on Σ. $[\![v]\!]_\Gamma := v|_{\Omega_1} - v|_{\Omega_2}$ stands for the jump of a function v across the interface Γ.

The existence and uniqueness of the weak solution to the problem (1)-(3) are well studied in the literature (see, e.g., [32]). The singularity and regularity of the solution with smooth μ, ε in polyhedral and smooth domains are considered in [21, 23]. The singularity of the solution of the Maxwell interface problems with polyhedral interfaces is studied in [22]. To the best of the authors' knowledge, the H^2 interface regularity of the solution to the Maxwell interface problem with smooth interfaces is missing in the literature. In this paper, we first prove the H^2 regularity of the solution to (1)-(3) in each subdomain Ω_1, Ω_2. Our proof is based on the

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$\boldsymbol{H}(\mathrm{curl})$-coercive Maxwell equations which is different from the $\boldsymbol{H}(\mathrm{curl}) \cap \boldsymbol{H}(\mathrm{div}\,)$-coercive Maxwell equations used in [21, 22, 23]. This new regularity result will be used in our finite element convergence analysis based on the Schatz argument in dealing with the indefiniteness of the time-harmonic Maxwell equations.

There exists a large literature on finite element methods for solving the time-harmonic Maxwell equations after the seminar work [45]. We refer to [32, 44] for the study of the $\boldsymbol{H}(\mathrm{curl})$-conforming h-methods, [25, 42] for the hp-methods, and [46, 36, 37] for the discontinuous Galerkin (DG) methods. The common assumptions in these studies are that the domains are polyhedral and the material interfaces are piecewise flat so that conforming tetrahedral or hexahedral meshes can be used. Much less studies have been devoted to finite element methods solving Maxwell equations on domains with curved boundary. We refer to [27, 33, 43] for body-fitted finite element methods, [11] for the isogeometric analysis, and [15] for the low order unfitted finite element method. We remark that the design of body-fitted high-order finite element methods depends on nonlinear element transforms from the reference element to the elements with curved boundary [6, 43]. It may be challenging to satisfy the conditions on the nonlinear element transforms which depend on the geometry of the interface or boundary in practical applications.

The original motivation of unfitted finite element methods in the DG framework is to release the time-consuming work of constructing shape regular meshes for domains with complex geometry. It turns out that the unfitted finite element methods also provide a natural way to design high-order methods without resorting to the nonlinear element transforms. Since the seminal work [31] for elliptic interface problems, considerable progress of the unfitted finite element methods has been made in the literature [13, 51, 39, 30, 4, 52, 38, 16]. The small cut cell problem, that is, the intersection of the domain and the elements can be arbitrarily small or anisotropic, can be solved by appropriate techniques of stabilization [13, 51, 39] or merging small cut cells with surrounding large elements [30, 4, 12, 16]. We refer to [16, 17] for further references and other approaches of unfitted finite element methods.

In [16] an adaptive high-order unfitted finite element method in two dimension is proposed for elliptic interface problems in which the hp a priori and a posteriori error estimates are derived based on novel hp domain inverse estimates and the concept of interface deviation. The interface deviation is a measure that quantifies the resolution of the geometry by the mesh. In [17], for any C^2 interface, a reliable algorithm is constructed to merge small interface elements with their surrounding elements to generate an induced mesh whose elements are large with respect to both domains, which solves the small cut cell problem. It is also shown in [17] that the exponential growth of the condition number of the stiffness matrix on the finite element approximation order, which is observed in the literature (e.g., [48, 17]), can be controlled by reducing the interface deviation. Therefore, arbitrarily high order unfitted finite element methods on automatically generated Cartesian meshes for solving elliptic interface problems can be achieved for arbitrarily shaped C^2 smooth interfaces.

The main purpose of this paper is to extend the high order unfitted finite element method for two-dimensional elliptic interface problems in [16] to solve the time-harmonic Maxwell equations (1)-(3). We characterize and quantify the mesh resolution of the geometry in two steps. We first introduce the concept of proper intersection of the interface and boundary to the elements and the concept of large element in three dimension, which allow us to show that each large element is a