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A FINITE VOLUME METHOD PRESERVING THE INVARIANT REGION PROPERTY FOR THE QUASIMONOTONE REACTION-DIFFUSION SYSTEMS

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Abstract. We present a finite volume method preserving the invariant region property (IRP) for the reaction-diffusion systems with quasimonotone functions, including nondecreasing, decreasing, and mixed quasimonotone systems. The diffusion terms and time derivatives are discretized using a finite volume method that satisfies the discrete maximum principle (DMP) and the backward Euler method, respectively. The discretization leads to an implicit and nonlinear scheme, and it is proved to preserve the invariant region property unconditionally. We construct an iterative algorithm and prove the invariant region property at each iteration step. Numerical examples are provided to confirm the accuracy and invariant region property of our scheme.

Key words. Reaction-diffusion systems, quasimonotone, nonlinear finite volume scheme, invariant region, distorted meshes, existence, model.

1. Introduction

Reaction-diffusion systems are mathematical models that describe the behaviors of a wide range of physical, biological, chemical, and electrical phenomena [1, 2, 11, 13, 14, 16, 23, 24]. They are used to simulate the variations in chemical substance concentrations caused by local reactions and diffusions in the field of chemistry, the spread of infectious diseases and population growth [5] in biology, the neutron diffusion theory and the Ginzburg-Landau equations for modeling superconductivity [6] in physics, and the FitzHugh-Nagumo model for simulating the transmission of electrical impulses in neurology, among others.

It is of great importance for the numerical methods to preserve the IRP. The IRP refers to the property of reaction-diffusion systems where the solution lies within the range of the initial and boundary values, reflecting the physical constraints of the unknown variables. Hence, the numerical solution is expected to preserve the IRP as well. Additionally, the IRP of numerical schemes is crucial to establishing a prior estimates, as well as existence and stability of the solution [22]. Proposing the IRP-preserving schemes for reaction-diffusion equations is necessary for both physical and mathematical aspects. The finite difference methods [3, 10, 12, 15] have been widely applied to solve the reaction-diffusion equations due to their simplicity. A fully implicit time-discretization method is employed in [12], where the IRP and stability of the scheme are established using M-matrix analysis. In [15], the authors use the exponential time differencing method and overlapping domain decomposition technique to develop a maximum bound principle (MBP) preserving method for one-component reaction-diffusion equations. MBP is considered as a specific form of invariant region. The nonstandard finite difference method together with a time-accurate and highly stable explicit method are combined in [3] to construct a positivity-preserving scheme for the reaction-diffusion model describing vegetation evolution in arid environments. The θ-weighted time-stepping scheme

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and corresponding iterative approach are developed in [10] to solve a class of semilinear parabolic equations. The discrete MBP is preserved under specified constraints on the time step and mesh size. However, most finite difference methods are restricted to rectangular meshes. Furthermore, the finite element method with implicit-explicit Euler time-discretization is utilized to solve 3D reaction-diffusion systems in [9], where the IRP is preserved on Delaunay triangular meshes. The nonlinear Galerkin method is employed in [17] to solve the system of reaction-diffusion equations. This approach requires to calculate the orthonormal basis for the space spanned by the eigenvectors of the diffusion operator. In [25], the finite volume method preserving the IRP is applied to a specific type of reaction-diffusion systems known as FitzHugh-Nagumo equation on polygonal meshes. A unified framework that covers many numerical schemes is established in [7] to yield an MBP-preserving method for semilinear parabolic equations.

The goal of this paper is to propose an IRP-preserving finite volume method solving coupled quasimonotone parabolic systems on distorted meshes. Compared to our previous work in [25], which could only handle specific nonlinear reaction terms, namely $f_1(u,v) = u(1-u)(u-a)-v$ and $f_2(u,v) = \rho u - \gamma v$, this work could handle more general nonlinear reaction terms. We employ the DMP-preserving finite volume scheme to discretize the spatial derivatives and utilize a fully implicit scheme to discretize the temporal derivatives. For the problems with three basic types of quasimonotone functions, we demonstrate that the implicit scheme is unconditionally IRP-preserving and has at least one solution. To solve the nonlinear scheme, we introduce a specific linear term into the iterative algorithm and prove the IRP preservation of the iterative method. Numerical experiments demonstrate that our scheme achieves second-order accuracy in the spatial direction and preserves the IRP for different problems. Additionally, we present a comparison with the nine-point scheme to demonstrate that the nine-point scheme fails to preserve the IRP.

This paper is organized as follows. Section 2 introduces the model problem and its corresponding invariant region theory. Section 3 presents the fully implicit finite volume scheme and analyze the preservation of the IRP. In Section 4, the iterative approach is described and its IRP is analyzed. In Section 5, we provide numerical experiments to demonstrate the accuracy and preservation of the IRP. Finally, Section 6 offers a summary of this paper.

2. Invariant region theory of the model problem

In this paper, we investigate the coupled system of two parabolic equations on a bounded space-time domain $Q_T = \Omega \times (0,T]$ as

$$\partial_t u - \nabla \cdot (\kappa_1 \nabla u) = f_1(u,v), \qquad \text{in } Q_T, \tag{1}$$

$$\partial_t v - \nabla \cdot (\kappa_2 \nabla v) = f_2(u,v), \qquad \text{in } Q_T, \tag{2}$$

subject to the initial conditions $u(\mathbf{x},0) = u_0(\mathbf{x})$ and $v(\mathbf{x},0) = v_0(\mathbf{x})$ on Ω and Dirichlet boundary conditions $u(\mathbf{x},t) = g_1(\mathbf{x},t)$ and $v(\mathbf{x},t) = g_2(\mathbf{x},t)$ on $S_T = \partial\Omega \times (0,T]$. Assume that Ω is an open bounded polygonal domain in $\mathbb{R}^2$, $\partial\Omega \in C^2$, κ_1 and κ_2 are coercive tensor-valued functions, and f_1 and f_2 are nonlinear functions of u and v.

The notations of standard Sobolev spaces are employed, with $(\cdot,\cdot)$ representing the $L^2(Q_T)$ inner product. Define bilinear forms $B_1(u,\phi_1) = (-u\partial_t\phi_1 + \kappa_1\nabla u, \nabla\phi_1)$ and $B_2(v,\phi_2) = (-v\partial_t\phi_2 + \kappa_2\nabla v, \nabla\phi_2)$. We say a pair of functions $(u,v) \in [W_2^{1,1}(Q_T) \bigcap L^\infty(Q_T)]^2$ is a weak solution to the problem (1)-(2) provided that