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FROM OBSTACLE PROBLEMS TO NEURAL INSIGHTS: FEEDFORWARD NEURAL NETWORK MODELING OF ICE THICKNESS

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Abstract. In this study, we integrate the established obstacle problem formulation from ice sheet modeling [1, 2] with cutting-edge deep learning methodologies to enhance ice thickness predictions, specifically targeting the Greenland ice sheet. By harmonizing the mathematical structure with an energy minimization framework tailored for neural network approximations, our method's efficacy is confirmed through both 1D and 2D numerical simulations. Utilizing the NSIDC-0092 dataset for Greenland [22] and incorporating bedrock topography for model pre-training, we register notable advances in prediction accuracy. Our research underscores the potent combination of traditional mathematical models and advanced computational techniques in delivering precise ice thickness estimations.

Key words. Neural networks, ice thickness estimation, obstacle problems, feedforward neural networks, mathematical modeling, partial differential equations.

1. Introduction

The melting of ice sheets, driven by climate change, is a topic of mounting concern across various scientific disciplines. This phenomenon is pivotal for understanding the dynamic processes of Earth's climate, particularly in regions such as Greenland. The melting of Greenland's ice sheet not only contributes to global sea level rise but also provides insights into intricate climate interactions and feedback loops. As such, mathematicians have developed complex models to delve deeper into ice sheet dynamics. Among these models, obstacle problems [3] offer a unique lens, presenting challenges in partial differential equations (PDEs). Over the years, numerous numerical methods have been devised to address these challenges. Most of these methods focus on providing approximation solutions to the weak variational inequality. Techniques like the Galerkin least squares finite element method ([5], [4], [6]), multigrid algorithm ([8], [7]), piecewise linear iterative algorithm [9], the first-order least-squares method [10], the level set method [11], and the dynamical functional particle method [12] have been employed with varying degrees of success.

In the wake of technological advancements, deep learning has emerged as a promising tool in many scientific applications. It has recently gained significant traction in solving differential equations and inverse problems ([13], [14], [15], [16], [17], [18]). Despite this momentum, its application to variational inequalities remains in its infancy. Some studies ([20], [21]) have innovatively applied deep learning to the traditional obstacle problem, whereas others [19] have ventured into using deep learning techniques for elliptic hemivariational inequalities. A common observation, however, is that many of these studies prioritize computations over theoretical insights.

With this backdrop, our paper endeavors to bridge this gap. We explore both traditional ice-sheet models [1, 2] and introduce a computational approach using

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deep learning to address the obstacle problem, deriving inspiration from its variational form. A central theme of our work is to discern the influence of parameters such as network size and training samples on the outcomes. Through rigorous numerical experiments, we substantiate the efficacy of our proposed method.

This article is organized as follows: Section 2 introduces the mathematical formulation of the model and elucidates the ice-thickness variational inequality. Section 3 sheds light on the energy minimization formulation. Section 4 delineates the approximation of the solution using fully connected feedforward deep neural networks, detailing its architecture, universality as an approximator, and the composite loss function tailored for optimal training and optimization. Section 5 showcases numerical experiments for one and two-dimensional problems, accompanied by solution visualizations and error analysis. Section 6 applies our model to data sourced from Greenland. We wrap up in Section 7, offering a concise summary of our study's principal insights and findings.

2. Mathematical Formulation of the Model

In this section, we present the mathematical formulation of the model, which is adapted from the work presented in [1, 2].

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, equipped with the standard Euclidean norm. A domain Ω in $\mathbb{R}^n$ is defined as a bounded and connected open subset of $\mathbb{R}^n$, whose boundary is Lipschitz continuous. Consider a subset Ω residing within $\mathbb{R}^2$. For any point $x = (x_1, x_2)$ contained within the closure of Ω, denoted as $\bar{\Omega}$, we will utilize common mathematical operators without going into their detailed definitions here.

The bedrock elevation is denoted by the function $b : \bar{\Omega} \to \mathbb{R}$. It's noteworthy that a positive value of b represents elevations above sea level, while negative values correspond to depths below the sea level.

Similarly, the elevation of the top surface of the ice sheet is characterized by the function $h : \bar{\Omega} \to \mathbb{R}$. It is imperative to emphasize that throughout the domain Ω, h always maintains a value greater than or equal to b. A visual representation of this relationship is provided in Figure 1. Consequently, the thickness of the ice, denoted as $H : h - b$, consistently remains nonnegative throughout $\bar{\Omega}$. This insight underscores the observation that studying changes in ice thickness is tantamount to addressing an obstacle problem, where the bedrock acts as the primary constraint.

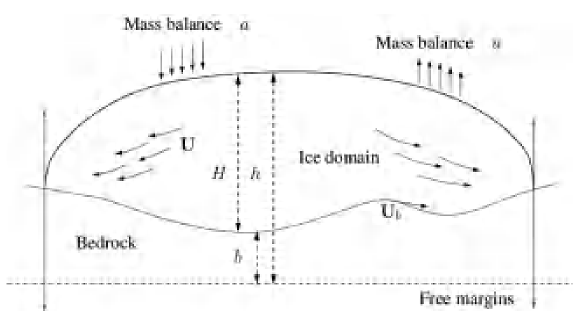

FIGURE 1. Cross-sectional view of an ice sheet with the respective notation, based on Jouvet et al. (2012).

This particular constraint implies the existence of a free boundary. Let's define

$$\Omega^+ = \{h > b\} = \{H > 0\} \tag{1}$$