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A DIFFERENCE VIRTUAL ELEMENT METHOD FOR THE 3D ELLIPTIC EQUATION WITH THE VARIABLE COEFFICIENT ON GENERAL CYLINDRICAL DOMAINS

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Abstract. In this paper, we present and analysis a difference virtual element method (DVEM) for the three dimensional (3D) elliptic equation on general cylindrical domains. This method combines the dimension splitting method and operator splitting technique to transform the virtual element solution of 3D elliptic equation into a series of virtual element solution of 2D elliptic equation based on (x, y) plane, where the central difference discretization is adopted in the z-direction. This allows us to solve partial differential equations on cylindrical domains at the low cost in mesh generation compared with 3D virtual element method. The H^1-norm error estimation of the DVEM is analysed in this paper. Finally, some numerical examples are performed to verify the theoretical predictions and showcase the efficiency of the proposed method.

Key words. 3D elliptic equation, difference virtual element, virtual element, cylindrical domain, error analysis.

1. Introduction

The development of the numerical methods for the 3D partial differential equation on general polygonal (polyhedral) meshes has been drawn considerable attention due to the extensive flexibility for the polygonal (polyhedral) meshes on the mesh generation, mesh deformation, fracture, combination, topology optimization, and mesh refinement and coarsening. In addition, the use of arbitrary-shape meshes can have good flexibility in dealing with complex data features. With regards to the spatial discretization, there exist many works devoted to treating the general polygonal and polyhedral elements. These methods include the finite volume method [1, 2], weak Galerkin finte element method, mimetic finite difference method [3] and the virtual element method (VEM) [4, 5, 6, 7].

The virtual element method was originally proposed in [8] to solve the Poisson equation and later has been successfully applied to a variety of partial differential equations such as convection diffusion equation, Allen-Cahn equation, Cahn Hilliard equation. More recently, mixed VEM was proposed for solving the fluid flow problems, see the Stokes problem [9, 10], Brinkman problem [11, 12], Stokes-Darcy problem [13, 14], Stokes complex in the VEM framework [15, 16], the magnetohydrodynamics problems [17] and the steady quasi-geostrophic equation of the ocean [18]. Several studies have contributed to the development and refinement of VEM for elliptic interface problems. For instance, Cao et al. [19] introduced immersed virtual element methods for two-dimensional elliptic interface problems. Chen et al. [20] focused on an interface-fitted mesh generator and virtual element methods for elliptic interface problems. Gómez et al. [21] explored space-time virtual elements for the heat equation. Their work extended the concept of VEM to evolve problems in time, which is particularly useful for capturing the temporal behavior of interfaces. Tushar et al. [22] investigated virtual element methods for general linear elliptic interface problems on polygonal meshes with small edges. Wang et

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al. [23] introduced a conforming virtual element method based on unfitted meshes for the elliptic interface problem.

The finite difference method, as an important numerical method, plays a crucial role in scientific calculating [24, 25, 26, 27]. However, the finite difference method is not easy to discretize the complex domain,especially in high dimensional space. Dimension splitting method [28, 29, 30, 31, 32, 30] and operator splitting method are two popular strategies to reduce the high dimensional problem into a series of low-dimensional problems. Based on the idea of dimension splitting method and operator splitting technique, He and Feng proposed the difference finite element method (DFEM) for solving 3D partial differential equations. In [33], the author used the DFEM based on P_1-P_1 conforming elements to solve the 3D Poisson e-quation and obtained the H^1 superconvergence results of this method by quadratic interpolation. In addition, Feng and his collaborators applied the DFEM to solve the 3D heat conduction equation [34] and obtained the H^1-superconvergence re-sults. Later in [35, 36, 37] , they proceeded the DFEM to solve the 3D continuous incompressible Stokes equations and Navier-Stokes equations and obtained the ex-istence, uniqueness and stability of the finite element solution as well as the optimal convergence.

For $L_3 > 0$, let $\Omega = \omega \times (0, L_3)$ where $\omega \subset \mathbb{R}^2$. We consider the following 3D elliptic equation with Dirichlet boundary condition:

$$
\begin{cases} -\widetilde{\nabla} \cdot (\widetilde{\mathbf{A}} \widetilde{\nabla} u) := -\partial_{zz} u - \nabla \cdot (\mathbf{A}(x,y) \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}
$$

where $\widetilde{\nabla} = (\partial_x, \partial_y, \partial_z) = (\nabla, \partial_z)$ and $\tilde{\mathbf{A}} \in [L^\infty(\Omega)]^{3\times 3}$ is the symmetric matrix-value function of form

$$
\widetilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A}(x,y) & 0 \\ 0 & 1 \end{pmatrix}.
$$

Here $\mathbf{A}(x,y) \in [L^\infty(\omega)]^{2\times 2} = \begin{pmatrix} A_{11}(x,y) & A_{12}(x,y) \\ A_{21}(x,y) & A_{22}(x,y) \end{pmatrix}$ is assumed to be uniformly elliptic and continuous in the sense that there exist two positive constants $0 < \alpha_* \leq \alpha^* < +\infty$ such that

$$
\alpha_* |\boldsymbol{\xi}|^2 \leq \sum_{i,j=1}^{2} A_{ij}(x,y) \xi_i \xi_j \leq \alpha^* |\boldsymbol{\xi}|^2 \quad \forall (x,y) \in \omega,
$$

for any $\boldsymbol{\xi} = (\xi_1, \xi_2)^\top \in \mathbb{R}^2$.

This manuscript introduces a novel DVE approach grounded in lower-order ele-ments for resolving 3D elliptic equations. By preserving the virtues of the virtual element method, the computational demands of intricate 3D scenarios are substan-tially reduced. The underlying principle of this DVE strategy involves utilizing finite difference method discretization in the z-direction to convert the 3D model into a collection of 2D elliptic equations, which are then approximated using the low-order virtual element method in the (x, y) plane. Consequently, the numerical solution to a complex 3D problem can be obtained by combining the numerical solutions of several 2D problems. Although the dimension of the coefficient matrix presented by the difference virtual element method remains unchanged, the stiffness matrix can be reused at each z-grid point without the need for reassembly, thereby conserving computational resources.

The structure of this article is as follows. In the next section, we introduce the virtual element space and the method of virtual elements in the 2D domain. In