INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 22, Number 3, Pages 307–339

doi: 10.4208/ijnam2025-1014

A LOCKING-FREE REDUCED-ORDER MODEL FOR SOLVING THE ELASTIC WAVE EQUATION

LU WANG, YOUCAI XU, AND MINFU FENG*

Abstract. In this paper, a new locking-free mixed full-order model (FOM) for solving the elastic wave equation is studied, and then a locking-free reduced-order model (ROM) based on the proper orthogonal decomposition (POD) technique is constructed, which greatly improves solving efficiency compared to FOM while maintaining the locking-free. Theoretical analysis of semi discrete and fully discrete schemes for the FOM and the ROM are also presented. Some numerical experiments verify the theoretical analysis results.

Key words. Elastic wave equation, mixed finite element method, proper orthogonal decomposition, locking free, implicit scheme, reduced-order model.

1. Introduction

The elastic wave equation, also known as the elastodynamic equation, can be used to simulate the propagation of waves in heterogeneous media and predict damage patterns caused by earthquakes [1]. To solve the elastic wave equation, there are many various numerical methods developed, such as the finite difference (FD) method [2] and the finite element (FE) method. Owing to the advantages of dealing with complex geometry and boundary conditions, the FE method is widely used for solving the elastic wave equation. Various FE methods have been developed, including conforming finite element methods [3, 4], spectral finite element methods [5], non-conforming finite element methods [6], and discontinuous Galerkin (DG) methods [7, 8, 9].

The elastic wave equation can be viewed as a nonstationary linear elasticity problem. The classical FE method poses two challenges when solving it. One is low computational efficiency, and the other is the locking phenomenon that occurs when $\lambda \to \infty$. Developing algorithms that can simultaneously overcome and improve both of the two problems is interesting.

In terms of improving computing efficiency, the ROM based on the POD method (POD-ROM) [10, 11, 12, 13] is an effective way. This method provides an orthogonal basis for describing a given dataset in the least-squares optimal sense, enabling us to determine the best low-dimensional approximation for a particular data collection. There have been successful attempts at POD-ROM for Navier-Stokes equation [14], Stokes equation [15], parabolic equations [16], and many other problems [17]. The results of such attempts have shown that ROM can significantly enhance computational efficiency while maintaining the accuracy of the models. Some research has also been carried out on ROM for wave equations, such as [18, 19]. As far as we know, there is little research on ROM for the elastic wave equation (1).

To make the ROM meet the locking-free property when solving (1), constructing a locking-free FOM is necessary. There are some methods to be locking-free, including the mixed finite element (MFE) method. Multiple types of MFE methods

Received by the editors on April 7, 2024 and, accepted on November 4, 2024.
2000 *Mathematics Subject Classification.* 35R35, 49J40, 60G40.
*Corresponding author.

have been developed for (1), such as the MFE method based on "stress displacement" [20, 21] and "stress velocity" [22, 23] mixed form. However, for the MFE methods mentioned above, the construction of the MFE space needs to satisfy the inf-sup condition and the construction of the stress space is complicated since the symmetry of the stress must be satisfied; otherwise, a new variable with weakly applied symmetry must be introduced.

In this paper, we construct an FOM using the MFE method of "displacement pseudo-pressure" by introducing pseudo-pressure $p = \lambda\nabla \cdot \boldsymbol{u}$, and we employ an unconditionally stable implicit scheme on time discretization. The constructed mixed form is similar to the Stokes problem; as a result, some stable MFE spaces that satisfy the inf-sup condition, applicable to the Stokes problem, can also be applied to (1) [24]. Then we construct a ROM based on the FOM using the POD technique and give analysis. The results show that the POD-ROM can improve efficiency compared with the FOM and keep locking-free.

Compared to the work of [25], our method differs in the following points. Firstly, the reference only considers the case $\mu = 0$, which is the acoustic wave equation, whereas we focus on the case $\mu \neq 0$. Secondly, the reference provides an error estimate of p that is dependent on $\lambda^{-1/2}$, we improved it in our analysis. Thirdly, the reference uses an explicit, fully discrete scheme, and we use the implicit scheme. Fourthly, we further propose a ROM to improve computational efficiency.

The remainder of this paper is structured as follows: Section 2 provides an overview of the elastic wave equation and the notions and conclusions used in the subsequent theoretical analysis. Section 3 constructs the FOM by the MFE method. In Section 4, we establish the POD-ROM and provide the algorithm. In Section 5, we validate the theoretical analysis of the FOM through some numerical tests. Section 6 provides a summary of the whole paper.

2. Preliminaries

This section presents some preliminaries of the elastic wave equation.

2.1. The mixed form of the elastic wave equation. In this section, we will discuss the mixed form of the elastic wave equation, along with some concepts and lemmas that will be used in the subsequent analysis.

Specifically, this paper will focus on the elastic wave equation in either two or three dimensions. Let $\Omega \subset \boldsymbol{R}^d, (d = 2, 3)$ denotes an open, bounded, connected domain with a Lipschitz continuous boundary $\partial\Omega$. We give a body exterior force $\boldsymbol{f}$, and the elastic wave model seeks a displacement vector field $\boldsymbol{u}(\boldsymbol{x}, t) = (u_x(\boldsymbol{x}, t), u_y(\boldsymbol{x}, t))$ in two dimensions (or $\boldsymbol{u}(\boldsymbol{x}, t) = (u_x(\boldsymbol{x}, t), u_y(\boldsymbol{x}, t), u_z(\boldsymbol{x}, t))$ in three dimensions) at time t that satisfies the following equation.

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \nabla \cdot \sigma(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

where $\sigma(\boldsymbol{u})$ is the symmetric stress tensor and ρ denotes the density for linear, homogeneous, and isotropic materials. For simplicity, we assume that the density ρ is a constant. The stress tensor $\sigma(\boldsymbol{u})$ is related to the strain tensor by Hooke's law

$$\sigma(\boldsymbol{u}) = 2\mu\varepsilon(\boldsymbol{u}) + \lambda(\nabla \cdot \boldsymbol{u})\mathbf{I}, \tag{2}$$

where the strain tensor $\varepsilon(\boldsymbol{u})$ is

$$\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \tag{3}$$