INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 22, Number 3, Pages 340–360

doi: 10.4208/ijnam2025-1015

NUMERICAL ANALYSIS OF AUGMENTED FVM FOR NONLINEAR TIME FRACTIONAL DEGENERATE PARABOLIC EQUATION

MUHAMMAD AAMIR ALI[1], ZHIYUE ZHANG[1,*] AND JIANQIANG XIE[2]

Abstract. Utilizing the nonlinear Time Fractional Degenerate Parabolic Equation (TFDPE) in modeling provides a comprehensive approach to studying phenomena exhibiting both fractional order dynamics and degenerate parabolic behavior, facilitating accurate predictions and insights across diverse scientific domains. However, the numerical solution of TFDPE is a challenging task and traditional numerical methods cannot solve this equation because of the spatial singularity influence. In this paper, we find the numerical solution of nonlinear TFDPE with both strongly and weakly degenerate cases using the higher order augmented finite volume method on uniform grids. To handle the singularity of TFDPE, we choose an intermediate point near the singular point and split the whole domain into singular and regular subdomains. Then, we find the solution on singular subdomain using the Puiseux series while on the regular subdomain we find the solution by finite volume schemes. The main idea is to recover the Puiseux series on singular subdomain using the Picard iteration methods which is also a challenging because of the time fractional derivative in the original equation. The solution on the singular subdomain is in the form of Puiseux series, which has multiple undetermined augmented variables and these variables play a role in organically combining the singular and regular subdomains. To approximate the time fractional derivative, we use the second order weighted and shifted Grünwald difference (WSGD) scheme and give the comparison of our results with L_1-scheme. We use the discrete energy method to prove that the schemes has temporal second order while spatial second and fourth-order on the whole domain and for the augmented variables in discrete L_2-norm. Finally, we give some numerical examples to confirm the accuracy and order of convergence of the proposed schemes for the whole domain and the augmented variables. We also give an interesting example with coefficient blow-up at the degenerate point and show the schemes are working the same as the other cases.

Key words. Time fractional degenerate parabolic equations, finite volume method, Puiseux series.

1. Introduction

In this paper, we establish second and fourth-order augmented finite volume schemes based on the Puiseux series for the following TFDPE

$$ {}_0^{\beta}D_t^C u + u_t - \left(x^{\alpha} u_x\right)_x = f\left(x,t,u\right), \quad (x,t) \in P, \tag{1}$$

where $0 < \alpha < 2$, $0 < \beta < 1$, $P = (0,b) \times (0,T)$ and ${}_0^{\beta}D_t^C$ is well known Caputo derivative [1,2]. The function $f(x,t,u)$ is a Lipchitz continuous at u, and it may have singularity at $x = 0$. The degenerate problem is where the coefficient of equation is nonnegative and the equation degenerates when the coefficient vanishes. It is obvious that the coefficient x^{α} of equation (1) vanishes when $x = 0$. Thus equation (1) is degenerate at part $\{0\} \times (0,T)$ of the lateral boundary. These mathematically results in a loss of uniform ellipticity, which in the real world would be represented such as tsunami wave velocity vanishing at the coastline, subsonic-sonic flow degenerating at the sonic state, local density of traffic flow at zero, etc.

Received by the editors on March 19, 2024 and, accepted on November 28, 2024.
2000 *Mathematics Subject Classification.* 35L05, 35L80, 65M08, 41A60, 68W30, 35R11.
*Corresponding author.

Thus, because of this degeneracy the traditional numerical methods cannot solve equation (1). To overcome this singularity, we use the spatial domain separation strategy. In this way, we choose an intermediate point near the singular point which is 0 for the equation (1) and convert into two subdomains namely singular subdomain P_s (left part) and regular subdomain P_r (right part). Then, we use the Puiseux series expansion on the singular subdomain to find the solution while on regular subdomain we use finite volume method on uniform grids without creating mesh points inside the singular subdomain (see, Figure 1). We also combined the both subdomains by the unknown variables of time involving in the singular subdomain solution.

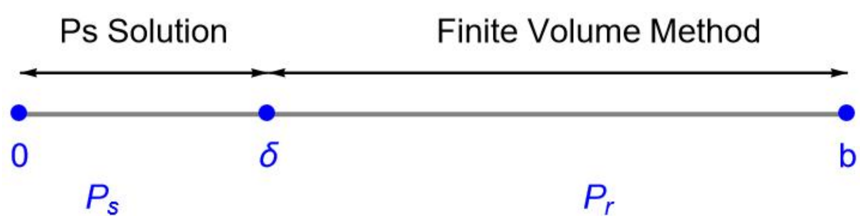

FIGURE 1. Domain separation strategy, where Ps solution means the Puisuex series solution.

The purpose of the term u_t in equation (1) is that we can approximate the time-fractional derivative directly by any second-order fractional approximation without modifying the original equation, and we can attain the numerical analysis of the schemes. Secondly, the equation (1) represents the models having both short and long-term memory effects because many systems have both short and long-term effects, such as the fractional Zener model and fractional Kelvin-Voigt model. Since the time-fractional derivative is for the long-term memory effects and the ordinary time derivative is for the short-term memory effects, we use both terms to get both kinds of memory effects from the single equation. On the basis of α, the equation (1) is classified as weakly and strongly degenerate. In [3], the authors solved equation (1) with classical time derivative only for $0 < \alpha < 1$, which is known as a weakly degenerate case, and they also created a mesh point in the singular subdomain, which may impact the accuracy and convergence of the scheme. It is also noted that the convergence analysis of the numerical schemes given in [3] is not provided. In this paper, we use the time derivative in the sense of fractional as well as solve the equation (1) for weakly degenerate ($0 < \alpha < 1$), strongly degenerate ($1 \leq \alpha < 2$) cases and ($\alpha < 0$) without creating mesh points inside the singular subdomain, due to which there is no impact on the accuracy and convergence of the schemes. The proposed method for this paper has no restriction in splitting the whole domain into the singular and regular subdomains, and we can choose the intermediate point independently. We also give the convergence analysis of the numerical schemes in L_2-norm using the energy method over the whole domain, which can support the convergence analysis for the schemes given in [3]. It is important to mention here that the schemes are also working well if we set $u_t = 0$ in equation (1), shown by examples. For equation (1), we have the following initial and boundary conditions:

$$u(0,t) = u(b,t) = 0 \text{ for } 0 < \alpha < 1, \tag{2}$$

$$(x^{\alpha} u_x)(0,t) = u(b,t) = 0 \text{ for } 1 \leq \alpha < 2, \tag{3}$$

$$u(x,0) = 0 \text{ for } 0 < \alpha < 2. \tag{4}$$