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A SELECTIVELY RELAXED ALTERNATING POSITIVE SEMIDEFINITE SPLITTING PRECONDITIONER FOR THE FLUX-LIMITED MULTI-GROUP RADIATION DIFFUSION EQUATIONS

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Abstract. In this article, we concentrate on the fast numerical computation of the radiation energy densities together with electron and ion temperatures of three-dimensional multi-group radiation diffusion equations, which is temporally discretized with the adaptive backward Eulerian scheme, linearized iteratively via the method of frozen coefficients and spatially approximated through a cell-centered finite volume discretization on the adaptive unstructured computational meshes. We present, analyze and implement an alternating positive semidefinite splitting preconditioning technique with two selective relaxations and algebraic multigrid subsolves, and provide an algebraic quasi-optimal selection approach to determine the involved parameters. Our parallel implementation is based on the software package jxpamg and the preconditioned flexible restarted generalized minimal residual solver has been examined by running realistic simulations of hydrodynamic instability on the Tianhe-2A supercomputer to demonstrate its numerical robustness, computational efficiency, parallel strong and weak scalabilities, and the competitiveness with some existing popular monolithic and block preconditioning strategies.

Key words. Radiation diffusion equations, alternating positive semidefinite splitting, selective relaxation, algebraic multigrid, parallel and distributed computing.

1. Introduction

On a spherically symmetrical bounded geometry, the flux-limited multi-group radiation diffusion (MGD) equations

(1)
$$\begin{cases} \dfrac{\partial E_g}{\partial t} = \nabla \cdot (D_g(E_g)\nabla E_g) + c(\sigma_{Bg}B_g(T_E) - \sigma_{Pg}E_g) + S_g, \ g = 1, \cdots, G, \\ \rho c_E \dfrac{\partial T_E}{\partial t} = \nabla \cdot (D_E(T_E)\nabla T_E) - c\displaystyle\sum_{g=1}^{G}(\sigma_{Bg}B_g(T_E) - \sigma_{Pg}E_g) + w_{IE}(T_I - T_E), \\ \rho c_I \dfrac{\partial T_I}{\partial t} = \nabla \cdot (D_I(T_I)\nabla T_I) - w_{IE}(T_I - T_E) \end{cases}$$

are the simplest and most extensively used approximation to the spatio-temporal orientation- and frequency-dependent thermal radiation transport equations, which compactly describe the propagations of high-energy photons in a physical system and the interactions with electrons directly and ions indirectly. It must be noticed that the thermal radiation transport process occurs in various branches of physics, such as the optical remote sensings, massive star formations and inertial confinement fusion experiments. The nonlinear PDE system (1) looks for the radiation energy density functions $E_1, \cdots, E_G$, the electron temperature function T_E and the ion temperature function T_I for some given density of medium ρ, the specific heat capacities c_E and c_I, the nonlinear radiation diffusion coefficient $D_g(E_g)$, the

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scattering and absorption coefficients σ_{Bg} and σ_{Pg}, the source item S_g and the electron scattering energy density $B_g(T_E)$ for the photon frequency group index $g = 1, \cdots, G$, the nonlinear thermal-conductivity coefficients $D_I(T_I)$ and $D_E(T_E)$ together with the energy transfer coefficient w_{IE}. In most situations, the analytic solution of problem (1) could not be available for arbitrary geometries and parameters or this nonlinear PDE system may not be directly solvable [33]. As a result, it needs to be discretized in the temporal dimension, at first, with the adaptive backward Eulerian scheme, yielding a series of semi-discrete nonlinear systems of the form

(2)
$$\begin{cases} -\nabla\cdot(D_g(E_g)\nabla E_g) + (\dfrac{1}{\Delta t_{k+1}} + c\sigma_{Pg})E_g - c\sigma_{Bg}B_g(T_E) = S_g + \dfrac{1}{\Delta t_{k+1}}E_g^{(k)}, \\ \qquad\qquad g = 1,\cdots,G, \\ -\nabla\cdot(D_E(T_E)\nabla T_E) + (\dfrac{\rho c_E}{\Delta t_{k+1}} + w_{IE})T_E + c\displaystyle\sum_{g=1}^{G}\sigma_{Bg}B_g(T_E) \\ \qquad\qquad - c\displaystyle\sum_{g=1}^{G}\sigma_{Pg}E_g - w_{IE}T_I = \dfrac{\rho c_E}{\Delta t_{k+1}}T_E^{(k)}, \\ -\nabla\cdot(D_I(T_I)\nabla T_I) + (\dfrac{\rho c_I}{\Delta t_{k+1}} + w_{IE})T_I - w_{IE}T_E = \dfrac{\rho c_I}{\Delta t_{k+1}}T_I^{(k)} \end{cases}$$

at the $(k+1)$-th time level, where $\Delta t_{k+1} = t_{k+1} - t_k$ is the actual time-step size and each continuous item with superscript (k) represents the correlative approximation at the preceding time level. Then, the nonlinear semi-discrete system (2) is linearized iteratively through the method of frozen coefficients [24], where the term $B_g(T_E)$ is approximated by its first-order Taylor series expansion

$$B_g(T_E) \approx B_g^{(\delta)} + \left(\frac{\partial B_g}{\partial T_E}\right)^{(\delta)}(T_E - T_E^{(\delta)})$$

due to its tanglesome nonlinearity while the others are replaced by their constant (0th-order) Taylor approximations at $E_g^{(\delta)}$, $T_I^{(\delta)}$ and $T_E^{(\delta)}$. We immediately obtain a sequence of coupled systems of second-order linear reaction-diffusion equations as follows

(3)
$$\begin{cases} -\nabla\cdot(D_g^{(\delta)}\nabla E_g) + (\dfrac{1}{\Delta t_{k+1}} + c\sigma_{Pg}^{(\delta)})E_g - c\sigma_{Bg}^{(\delta)}\left(\dfrac{\partial B_g}{\partial T_E}\right)^{(\delta)}T_E \\ \qquad = S_g^{(\delta)} + \dfrac{1}{\Delta t_{k+1}}E_g^{(k)} + c\sigma_{Bg}^{(\delta)}\left[B_g^{(\delta)} - \left(\dfrac{\partial B_g}{\partial T_E}\right)^{(\delta)}T_E^{(\delta)}\right],\ g = 1,\cdots,G, \\ -\nabla\cdot(D_E^{(\delta)}\nabla T_E) + \left[\dfrac{\rho c_E^{(\delta)}}{\Delta t_{k+1}} + w_{IE}^{(\delta)} + \displaystyle\sum_{g=1}^{G}c\sigma_{Bg}^{(\delta)}\left(\dfrac{\partial B_g}{\partial T_E}\right)^{(\delta)}\right]T_E \\ \qquad -\displaystyle\sum_{g=1}^{G}c\sigma_{Pg}^{(\delta)}E_g - w_{IE}^{(\delta)}T_I = \dfrac{\rho c_E^{(\delta)}}{\Delta t_{k+1}}T_E^{(k)} - \displaystyle\sum_{g=1}^{G}c\sigma_{Bg}^{(\delta)}\left[B_g^{(\delta)} - \left(\dfrac{\partial B_g}{\partial T_E}\right)^{(\delta)}T_E^{(\delta)}\right], \\ -\nabla\cdot(D_I^{(\delta)}\nabla T_I) + (\dfrac{\rho c_I^{(\delta)}}{\Delta t_{k+1}} + w_{IE}^{(\delta)})T_I - w_{IE}^{(\delta)}T_E = \dfrac{\rho c_I^{(\delta)}}{\Delta t_{k+1}}T_I^{(k)}, \end{cases}$$