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A CELL-CENTERED FINITE ELEMENT METHOD WITH IMPOSED FLUX CONTINUITY AND STREAMLINE UPWIND TECHNIQUE FOR ADVECTION-DIFFUSION PROBLEMS ON GENERAL MESHES

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Abstract. We extend the cell-centered finite element method (CCFE) [1] with imposed flux continuity and streamline upwind technique to solve the advection-diffusion equations with anisotropic and heterogeneous diffusivity and a convection-dominated regime on general meshes. The scheme is cell-centered in the sense that the solution is computed by cell unknowns of the primal mesh. From general meshes, the method is constructed by the dual meshes and their triangular submeshes. The scheme gives auxiliary edge unknowns interpolated by the multipoint fluid approximation technique to obtain the local continuity of numerical fluxes across the interfaces. In addition, the scheme uses piecewise linear functions combined with a streamline upwind technique on the dual submesh in order to stabilize the numerical solutions and eliminate the spurious oscillations. The coercivity, the strong and dual consistency, and the convergence properties of this method are shown in the rigorous theoretical framework. Numerical results are carried out to highlight accuracy and computational cost.

Key words. Advection-diffusion equations, anisotropic and heterogeneous diffusion, convection-dominated regime, cell-centered schemes, and convergence analysis.

1. Introduction

Many mathematical models involving Partial Differential Equations (PDEs) with both advection and diffusion terms play a fundamental role in solving complicated problems such as various fluid flow, Navier-Stokes equations, etc. The advection-diffusion problems, determined by two physical mechanisms: advection and diffusion, still pose many challenges in finding numerical solutions, especially when the diffusion is anisotropic (*e.g.* tensor-valued) and heterogeneous (*e.g.* nonsmooth, possibly with discontinuities) combined with strongly dominant convection.

On the one hand, one can hardly obtain the approximate solution which converges to the weak one for some general problems with a heterogeneous and anisotropic tensor, possibly with large discontinuities. In fact, when it comes to the discontinuous diffusion problems with the convective term, their approximate solutions computed by the standard finite element method (FEM) can be inaccurate [2]. The authors of [3] proposed a cell-centered scheme, *e.g.* the standard finite volume method (FVM), to address this issue; however, it requires admissible meshes as computational grids [4]. The multi-point flux approximation methods (MPFA) that are also cell-centered schemes precisely approximate the solutions by imposing the local conservation of fluxes [5, 6]. Nevertheless, the MPFA methods only satisfy the coercivity under suitable conditions on the mesh and the permeability tensor. In [7], the authors represented a MUSCL-like cell-centered finite volume method to

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approximate the solution of multi-dimensional steady advection-diffusion equations, but only for the diffusion process driven by the scalar viscosity field in $C^1(\overline{\Omega})$, not a tensor (possibly with discontinuous). Furthermore, a variety of efficient hybrid numerical schemes have been developed in the last decade to approximate solutions of diffusive equations on general grids, for example, the hybrid mimetic (HM) method [8, 9], the discontinuous Galerkin (DG) method [10, 11], the mimetic finite-difference (MFD) method [12], the mixed finite volume (MFV) method [13], the hybrid finite volume (HFV) method [8], and the discrete duality finite volume method (DDFV) [14]. However, these hybrid methods must depend on more than two different types of unknowns including edges, vertices, and cell ones. This can result in much greater computational cost in the implementation. Therefore, these methods need to rely on condensation arguments (such as Schur complement reduction [15], domain decomposition [16]) in order to reduce the size of the linear systems that need to be solved.

On the other hand, when the problems are isotropic diffusion and convection-dominated, their solutions possess interior and boundary layers. These boundary layers are small subregions where the derivatives of the solution are very large. The widths of these layers are usually significantly smaller than the mesh size, which means the layers are not properly resolved. This leads to unwanted spurious (nonphysical) oscillations in the numerical solution analyzed in [17]. The classical Galerkin formulation is inappropriate for the advection-diffusion problems since, in the case of dominant convection, the discrete solution is usually globally polluted by spurious oscillations, which causes a severe loss of accuracy and stability. To overcome this challenge, according to [9], there are two possible approaches for the convection-diffusion problems. The first approach is that the diffusive term is approximated, and then some forms of centered or upwind approximation of the convection term are implemented to discretize the boundary problems [18, 19, 20]. For the second approach, the total flux of both diffusive and convective terms is approximated, which seems more popular in the finite-element practitioner community. Due to the stability properties and higher-order accuracy, [21] commented that the streamline upwind/ Petrov-Galerkin (SUPG) method developed by Brooks and Hughes [22] is regarded as one of the most efficient procedures for solving convection-dominated equations.

In addition, for the convection-diffusion-reactions equations with a symmetric and uniformly positive definite dispersion-diffusion matrix, the finite volume element methods considered in [23] are based on a Petrov-Galerkin formulation in which the solution space consists of continuous piecewise polynomial functions and the test space consists of piecewise constant functions. This choice of test space is essential for preserving the local conservation property of the method. However, these methods are only implemented on triangular primal meshes, since their piecewise linear finite element spaces are defined on triangulations of the domain; furthermore, they are not cell-centered schemes.

In this paper, we propose a numerical method, namely cell-centered finite element (CCFE), to solve the advection-diffusion problems. This results in addressing two aforementioned challenges due to several following advantages:

1. In the case of heterogeneous and anisotropic diffusivity, possibly with discontinuities, the scheme uses a first-order finite approximation space for the solution and multi-point flux approximations for the discrete gradients to satisfy the local continuity of fluxes. In addition, the scheme adds a streamline upwind diffusion term developed by Brooks and Hughes [22] to