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A CONFORMING DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD FOR SECOND-ORDER PARABOLIC EQUATION

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Abstract. The conforming discontinuous Galerkin (CDG) finite element method is an innovative and effective numerical approach to solve partial differential equations. The CDG method is based on the weak Galerkin (WG) finite element method, and removes the stabilizer in the numerical scheme. And the CDG method uses the average of the interior function to replace the value of the boundary function in the standard WG method. The integration by parts is used to construct the discrete weak gradient operator in the CDG method. This paper uses the CDG method to solve the parabolic equation. Firstly, the semi-discrete and full-discrete numerical schemes of the parabolic equation and the well-posedness of the numerical methods are presented. Then, the corresponding error equations for both numerical schemes are established, and the optimal order error estimates of H^1 and L^2 are provided, respectively. Finally, the numerical results of the CDG method are verified.

Key words. Conforming discontinuous Galerkin finite element method, parabolic equation, weak Galerkin finite element method, optimal order convergence.

1. Introduction

The parabolic equation is an essential class of equations in partial differential equations. Its unique concept and properties make it play a huge role in physics and mathematics and have significant theoretical value. Many problems can be described by parabolic equations in life, for example, heat conduction of objects, flow problems of porous media, and diffusion problems of pollutant concentration. It is challenging to obtain analytical solutions on these practical problems, so scholars began to study their numerical solutions, which provides a solid theoretical basis for solving practical problems.

In this paper, we consider the initial-boundary value problems for second-order parabolic equation: Find u satisfies

$$(1)\qquad \begin{cases} u_t - \nabla\cdot(a\nabla u) &= f, \quad \mathbf{x}\in\Omega, \quad t\in J, \\ u &= 0, \quad \mathbf{x}\in\partial\Omega, \, t\in J, \\ u(\cdot,0) &= \psi, \quad \mathbf{x}\in\Omega, \end{cases}$$

where $J = \left[0,\overline{T}\right]$, $\overline{T} > 0$, $\Omega\subset\mathbb{R}^2$ is a polygon domain, and the boundary $\partial\Omega$ is Lipschitz continuous. And the source term $f(x,t)\in L^\infty\left(0,\overline{T};L^2(\Omega)\right)$ and the initial value $\psi\in H^2(\Omega)$. Assume that $a(\cdot)_{2\times 2}\in[L^\infty(\Omega)]^{2\times 2}$ is a symmetric matrix-valued function, which satisfies

$$C_1\eta^T\eta \le \eta^T a\eta \le C_2\eta^T\eta, \quad \forall\eta\in\mathbb{R}^2,$$

here C_1 and C_2 are two positive constants with $0 < C_1 < C_2 \ll \infty$.

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The variational formulation of the parabolic equation (1) is to find $u \in L^2\Big(0, \overline{T}; [H_0^1(\Omega)]^d\Big)$, such that

$$
\left\{ \begin{array}{rcl} (u_t, v) + (a\nabla u, \nabla v) &=& (f, v), \quad \forall\, v \in H_0^1(\Omega),\, t \in J, \\ u(\cdot, 0) &=& \psi. \end{array} \right. \tag{2}
$$

The Sobolev spaces are defined as follows:

$$
H^1(\Omega) := \left\{ v | v \in L^2(\Omega), \nabla v \in [L^2(\Omega)]^2 \right\},
$$

$$
H_0^1(\Omega) := \left\{ v | v \in H^1(\Omega), v|_{\partial\Omega} = 0 \right\},
$$

$$
L^2(0, \overline{T}; V) := \left\{ v \,|\, v(\cdot, t) \in V, \forall t \in \left[0, \overline{T}\right], \int_0^{\overline{T}} \|v(\cdot, t)\|_V^2 dt < \infty \right\},
$$

here V is a Sobolev space with a norm $\|\cdot\|_V$.

There are many numerical methods to solve the parabolic equation, such as the finite element method (FEM) [14, 33], the nonconforming finite element method (NC-FEM) [31], the discontinuous Galerkin (DG) finite element method [3, 13], the virtual element method [15, 32], the weak Galerkin (WG) finite element method [1, 4, 5, 21], etc. In this paper, we propose a conforming discontinuous Galerkin (CDG) finite element method to solve the parabolic equation.

The CDG method is based on the WG method [2, 6, 7, 16, 17, 22]. Its main idea is to use the discontinuous polynomial as the approximate function and increase the degree of the polynomial for calculating the weak differential operators. Using higher-order degree polynomials can effectively ensure the weak continuity of discontinuous functions over element boundaries and substantially reduce computational complexity without altering the dimensions of the stiffness matrix and the global sparsity. In contrast to the WG method, the CDG method uses the averages of the interior functions to replace the boundary functions, reducing the number of boundary degrees of freedom. It has the advantages of being easy to construct the finite element space and the numerical scheme. In addition, the CDG numerical scheme is amenable to parallel computing, thereby effectively mitigating the computational overhead. Recently, the CDG method has garnered considerable scholarly attention and has been successfully used to solve the second-order elliptic problems [25–27], Stokes problems [10, 28], Biharmonic problems [29, 30], elliptic interface problems [23], linear elasticity interface problems [24], and so on.

In this paper, we use the CDG method to solve the initial-boundary value problems for second-order parabolic equation. In the CDG scheme, the approximation of the function is achieved through the employment of the discontinuous k-th degree polynomial. Concomitantly, the stabilizer terms within the numerical method are eliminated by increasing the polynomial degree for calculating the weak differential operators. The numerical schemes are presented for the semi-discrete spatial case, wherein only space is discretized, and the full-discrete case, which involves the discretization of time and space. Subsequently, the error equations for semi-discrete and full-discrete schemes are presented. Additionally, optimal order error estimates in the H^1 and L^2 norms are derived.

An outline of this paper is as follows. In Section 2, we propose a semi-discrete CDG scheme for the parabolic equation (1). In Section 3, the full-discrete CDG scheme for the parabolic equation (1) is established. In Section 4, we derive the optimal order error estimate for the semi-discrete CDG scheme and full-discrete CDG scheme. In Section 5, numerical results are presented to validate the accuracy