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MEAN SQUARE STABILITY OF NUMERICAL METHOD FOR STOCHASTIC VOLTERRA INTEGRAL EQUATIONS WITH DOUBLE WEAKLY SINGULAR KERNELS

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Abstract. The main goal of this paper is to develop an improved stochastic θ-scheme as a numerical method for stochastic Volterra integral equations (SVIEs) with double weakly singular kernels and demonstrate that the stability of the proposed scheme is affected by the kernel parameters. To overcome the low computational efficiency of the stochastic θ-scheme, we employed the sum-of-exponentials (SOE) approximation. Then, the mean square stability of the proposed scheme with respect to a convolution test equation is studied. Additionally, based on the stability conditions and the explicit structure of the stability matrices, analytical and numerical stability regions are plotted and compared with the split-step θ-method and the θ-Milstein method. The results confirm that our approach aligns significantly with the expected physical interpretations.

Key words. Stochastic Volterra integral equations, weakly singular kernels, stochastic θ-scheme, SOE approximation, mean square stability.

1. Introduction

It is well known that a stochastic Volterra integral equation (SVIE) may be generalized from a stochastic differential equation (SDE) [28]-[33] or a standard Volterra integral equation [4, 16, 17, 42]. Equations of these kinds are often used in many scientific fields. For instance, they are often employed in the modelling of biological systems [18], financial markets [9, 13], control science [3], engineering [8] and health care [15]. Since a closed-form solution for SVIEs is generally not available, stochastic numerical schemes offer reliable techniques for studying the behavior of solutions [39, 45, 46, 50, 51]. Therefore, stochastic numerical analysis is a special area of interest in the study of SVIEs [1]. Moreover, the stochastic integral term in the equation commonly lacks a martingale property, leading to a non-Markovian process for the equation's solution. This non-Markovian behavior arises due to the dependence of SVIE kernels on the variable t. As a result, SVIEs are more demanding to analyze and solve in some ways than SDEs, and the accompanying numerical analysis is likewise more challenging. SVIEs with smooth kernels are now supported by the majority of numerical approximations. For example, Xiao et al. [6] presented a collocation method with split-step for SVIEs. Liang et al. [25] presented the Euler–Maruyama method for linear SVIEs of strong convergence with order $\frac{1}{2}$. In [24], the authors presented the modified stochastic θ-methods for the numerical integration of SVIEs.

However, fractional Brownian motion research and the discussion of specific problems in the area of stochastic partial differential equations (SPDEs) are two areas where SVIEs with singular kernels can be identified [44]. The singularity of the kernel at both limits of integral poses the biggest obstacle to implementing stochastic numerical approximations. Zhang [48] peruse the convergence of Euler Maruyama (EM) scheme with the property of large deviations for SVIEs with singular kernel. Xiao [49] find out the convergence order for SVIEs under the EM method with

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the feature of Abel-type kernels. Wang [47] proved an existence and uniqueness theorem under non-Lipschitz, linear growth condition, and a few integrable conditions. In particular, Li et al. [19] offered two numerical schemes for solving weakly singular SVIEs and extracted strong convergence rates for both of them.

Due to the singularity of the integrand, both integral limits present additional challenges. Unfortunately, the essential Itô formula, a powerful tool in studying SDEs, is not accessible in this research. Thus, we must embark on a search for alternative methods. For instance, Li et al. [41] used the Gronwall inequality to analyze the precise asymptotic separation rate of two alternative double singular SVIE solutions, each with two different initial data. In order to solve the SVIEs with double singular kernels, Li et al. [5] proposed the Galerkin approximation and proved strong convergence rates. Dai and Xiao [38] take into account nonlinear SVIEs with double weakly singular kernels. The d-dimensional SVIE with double weakly singular kernels is taken into consideration in this study as:

(1)

$$y(t) = y_0+\int_0^t (t-s)^{-\gamma_1} s^{-\sigma_1} F(y(s))ds+\int_0^t (t-s)^{-\gamma_2} s^{-\sigma_2} G(y(s))dW(s), \quad t \in [0,T],$$

where

- The functions $F : \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}^d \to \mathbb{R}^{d\times r}$ are nonlinear Borel measurable, and the initial data $y_0 \in \mathbb{R}^d$ satisfies $\mathbb{E}\big(|y_0|^2\big) < +\infty$.
- The r-dimensional Wiener process $\{W(t)\}_{t\in[0,T]}$ is defined on a complete filtered probability space.
- For $i = 1, 2$, the parameters γ_i and σ_i are non-negative, satisfying $0 < \gamma_1 + \sigma_1 < 1$, and $0 < \gamma_2 + \sigma_2 < \frac{1}{2}$.

For SVIEs (1) no results have been found regarding the stability of the analytical solution. Therefore, we conclude that by utilizing the SOE approximation, we analyze the mean square stability of the suggested scheme. However, numerical approximations for SVIEs with double singular kernels require the storage and utilization of the entire solution history throughout the calculation process. This significantly increases the computational and memory costs. For instance, the EM method has an average storage requirement of $\mathcal{O}(N)$ and an overall calculation cost of $\mathcal{O}(N^2)$. Therefore, it is crucial to find ways to reduce the computational costs. The aim behind the SOE approximation, which is used to approximate the kernel functions, is to reduce the computational complexity and memory cost of the numerical schemes [7, 10]. Hairer et al. [35] used fast Fourier transform techniques and the convolution structure to quickly solve nonlinear Volterra convolution equations. In order to overcome the poor computing efficiency, the SOE approximation was used by Dai and Xiao [38] to offer a fast EM method for double singular kernels in Levy-driven SVIEs. Wang et al. [20] proved the stability and convergence analysis of fast θ-Maruyama scheme for SVIEs of convolution type. Additionally, stability analysis of numerical methods for SVIEs, including the split-step θ-method [4], the θ-method [23] and the improved stochastic θ-method [24] has been thoroughly studied. Furthermore, the convergence analysis is the primary focus of the research on numerical approaches for SVIEs. We are aware of very little information regarding the analytical and numerical stability characteristics of SVIEs with singular kernels. To duplicate the mean square stability of the analytical solution, Doan et al. [40] developed an exponential EM method. Tuan [26] investigated the asymptotic mean square stability of solutions to SVIEs driven by a multiplicative white noise. As a result, by utilizing the SOE approximation, the scheme outlined in