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A CLASS OF RUNGE-KUTTA METHODS FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a class of Runge-Kutta (RK) methods for backward stochastic differential equations (BSDEs). The convergence rate is studied and the corresponding order conditions are obtained. For the conditional expectations involved in the methods, we design an approximation algorithm by combining the characteristics of the methods and replacing the increments of Brownian motion with appropriate discrete random variables. An important feature of our approximation algorithm is that interpolation operations can be avoided. The numerical results of four examples are presented to show that our RK methods provide a good approach for solving the BSDEs.

Key words. Backward stochastic differential equations, Runge-Kutta methods, order condition, conditional expectation.

1. Introduction

Consider the backward stochastic differential equation (BSDE) of the integral form

$$y(t) = \varphi(W(T)) + \int_t^T f(s, y(s), z(s))\, ds - \int_t^T z(s)\, dW(s),\ t \in [0, T], \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times m} \longrightarrow \mathbb{R}^d$ is a Lipschitz-continuous function, $W(t) = (W^1(t), W^2(t), \cdots, W^m(t))$ is an m-dimensional Wiener process supported by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}\}_{0\le t\le T})$, and the function $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^d$ has continuous and bounded first derivatives.

The existence and uniqueness of the solution of (1) was first proved in [12]. Moreover, by [13] and [14], we know that the solution of (1) can be rewritten as

$$y(t) = u(t, W(t)),\ z(t) = \nabla u(t, W(t)),\ t \in [0, T], \tag{2}$$

where ∇u is the gradient of $u(t, x)$ with respect to the variable x, and $u(t, x)$ is the solution of the terminal value Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\displaystyle\sum_{i=1}^m \dfrac{\partial^2 u}{\partial x_i^2} + f(t, u, \nabla u) = 0,\ t \in [0, T),\ x \in \mathbb{R}^m, \\ u(T, x) = \varphi(x),\ x \in \mathbb{R}^m. \end{cases} \tag{3}$$

The smoothness of the solution u depends on the smoothness of the functions f and φ (see, e.g., [6, 24]). Specifically, if $f \in C_b^{k,2k,2k},\ \varphi \in C_b^{2k+\epsilon},\ k \in \mathbb{Z}^+,\ \epsilon \in (0, 1)$, then we have $u \in C_b^{k,2k}$, where $C_b^{k,2k,2k}$ denotes the set of continuously differentiable functions $\phi(t, y, z)$ with uniformly bounded partial derivatives $\partial_t^{l_0}\partial_y^{l_1}\partial_z^{l_2}\phi$ for $2l_0 + l_1 + l_2 \le 2k$, $C_b^{k,2k}$ denotes the set of functions $\phi(t, x)$ with uniformly bounded partial derivatives $\partial_t^{l_0}\partial_x^{l_1}\phi$ for $2l_0 + l_1 \le 2k$, and $C_b^{2k+\epsilon}$ denotes the set of functions $\phi(x)$ such that $\partial_x^l\phi,\ l \le 2k$ are uniformly bounded and $\partial_x^{2k}\phi$ is Hölder continuous with index ϵ.

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As is well known, it is very difficult to find the analytic solution to most BSDEs. Therefore, developing numerical methods for solving BSDEs is becoming highly desired in practical applications. Up to now, many works on numerical methods of the BSDEs or their extensions forward-backward stochastic differential equations (FBSDEs) have been done. The methods in [5, 8, 9, 10, 11] are developed based on the relation between the BSDEs and the corresponding Cauchy problem. The methods in [3, 15, 16, 19, 20] are developed directly based on the BSDEs.

In recent years, there has been much interest in developing numerical methods based directly on the BSDEs. In particular, the linear multistep methods for solving ordinary differential equations (ODEs) have been successfully extended to solving BSDEs (see, e.g., [1, 17, 18, 21, 23, 25]). However, Runge-Kutta (RK) methods, as another type of important numerical methods for the ODEs, are rarely used to solve the BSDEs. As far as we know, there are currently only two references that have studied RK methods for the BSDEs [2, 4]. The authors of [4] studied a specific second-order RK method. The authors of [2] introduced a class of RK methods and provided rigorous convergence analysis results.

In the present paper, we will introduce a class of RK methods for the BSDEs (1). The order conditions up to third order are obtained for our RK methods. Based on the order conditions, we give two specific explicit RK methods. Combining the characteristics of our RK methods and replacing the increments of Brownian motion with some appropriate discrete random variables, we design an approximation algorithm for the conditional expectations involved in the RK methods. Our RK methods is different from the RK methods proposed in [2]. The main difference lies in the calculation of the internal stages about variable z (see method (7)), which is more conducive to design the approximation algorithm for the conditional expectations (see Remark 1). In addition, no interpolation operations are required for our approximation algorithm of the conditional expectations. What's more, the ideal of our approximate algorithm can be applied to many other methods for solving the BSDEs (see below).

This paper is organized as follows. In section 2, we introduce our RK methods. We study the convergence rate and obtain the corresponding order conditions in section 3. In section 4, the approximation algorithm for the conditional expectations is presented. Finally, we present some numerical results to verify our theoretical results.

2. RK methods for the BSDEs

Under the uniform time stepsize $h = \frac{T}{N},\ t_n = nh,\ n = 0, 1, 2, \cdots, N$ (N is a given positive integer), we have

$$(4)\quad \begin{cases} y(t_n) = y(t_{n+1}) + \int_{t_n}^{t_{n+1}} f(s, y(s), z(s))\ ds - \int_{t_n}^{t_{n+1}} z(s)\ dW(s),\ n < N, \\ y(t_N) = \varphi(W(T)). \end{cases}$$

Inspired by [19, 20], for equation (4), we can establish the following two ordinary differential reference equations

$$(5)\qquad y(t_n) = \mathbb{E}_{t_n}\Big[y(t_{n+1}) + \int_{t_n}^{t_{n+1}} f(s, y(s), z(s))\ ds\Big],$$