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A FINITE DIFFERENCE METHOD FOR AN INTERFACE PROBLEM WITH A NONLINEAR JUMP CONDITION

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Abstract. We propose a finite difference approach to numerically solve an interface heat equation in one dimension with discontinuous conductivity and nonlinear interface condition. The discontinuous physical solution is sought among the multiple solutions of the nonlinear equation. Our method finds the approximate jump of the exact solution by two auxiliary linear problems with finite jumps. The approximate physical solution is then obtained by a weighted sum. The convergence and stability of the method are analyzed by the method of nonnegative matrices. Numerical examples are given to confirm the theory. In particular, numerical simulations are demonstrated in regards to the study of polymetric ion-selective electrodes and ion sensors.

Key words. Finite difference method, nonlinear parabolic problems, ion sensors, transmission equation.

1. Introduction

In this paper, we study the numerics associated with the following nonlinear interface parabolic problem:

(NIPP)

Find $u : \Omega^- \cup \Omega^+ \times [0,T] \to \mathbb{R}$ such that $u = u(x,t)$ satisfies

$$\mathcal{L}u := u_t - (\beta u_x)_x + qu = f, \quad x \in \Omega^- \cup \Omega^+, t \in (0,T] \tag{1}$$

$$u(a) = \xi(t), \quad u(b) = \eta(t), \tag{2}$$

$$[u]_\alpha = \lambda u^+ u^-, \tag{3}$$

$$[\beta u_x]_\alpha = 0, \tag{4}$$

$$u(x,0) = g(x), \quad x \in \Omega^- \cup \Omega^+. \tag{5}$$

Here α is a fixed interface point, $\Omega^- = (a,\alpha), \Omega^+ = (\alpha,b)$, the coefficient $\beta = \beta(x) > 0$ is piecewise constant:

$$\beta = \begin{cases} \beta^- & \text{on } \Omega^-, \\ \beta^+ & \text{on } \Omega^+, \end{cases} \tag{6}$$

the functions $q = q(x) \geq 0$ and $f = f(x,t)$ are assumed to be sufficiently smooth so that the solution $u(\cdot,t)$ is smooth in $\Omega^- \cup \Omega^+$ for all $t \in (0,T]$. This assumption is needed since we consider finite difference methods throughout the paper. Also note that the boundary conditions (2) are allowed to be time dependent. The parabolic problem under consideration is nonlinear when $\lambda \neq 0$ due to the interface jump condition (3) in which the jump

$$[u]_\alpha = u^+ - u^-, u^+ = \lim_{x\to\alpha^+} u(x,t), u^- = \lim_{x\to\alpha^-} u(x,t)$$

is proportional to u^+u^- with a proportionality constant λ. In (4), $[\beta u_x]_\alpha$, the jump in flux βu_x, is assumed to be zero. Problem NIPP (1)–(5) is motivated by [4] in which Hetzer and Meir presented an idealized mathematical model in the study

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of polymetric membrane, ion-selective electrodes and ion sensors. Following [4], a schematic diagram is shown in Figure 1 where u is the concentration, $a = -a_{aq}$, $b = a_{org}, \Omega^- = I_{aq}, \Omega^+ = I_{org}$. See [4] for more details and the finite element simulations associated with the model. A review paper on how pulsed amperometric sensors work is [2]. However, for ease of reference we will also call the problem NIPP the heat equation with discontinuous conductivity and nonlinear interface jump.

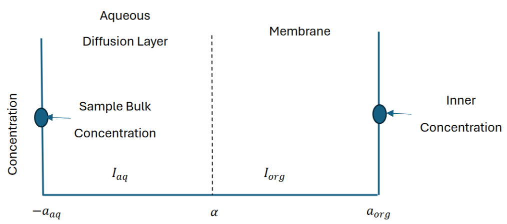

FIGURE 1. Polymeric Membrane, Ion-selective electrode, and Ion Sensor.

The $s-$parameter method. The solutions of NIPP could be obtained as follows. Let u_0 be the solution of

$$\mathcal{L}u = f, \quad x \in \Omega^- \cup \Omega^+, t \in (0, T] \tag{7}$$

$$u(a) = \xi(t), \quad u(b) = \eta(t), \tag{8}$$

$$[u]_\alpha = 0, \tag{9}$$

$$[\beta u_x]_\alpha = 0, \tag{10}$$

$$u(x, 0) = g(x), \quad x \in \Omega^- \cup \Omega^+. \tag{11}$$

On the other hand, let u_1 be the solution of

$$\mathcal{L}u = 0, \quad x \in \Omega^- \cup \Omega^+, t \in (0, T] \tag{12}$$

$$u(a) = 0, \quad u(b) = 0, \tag{13}$$

$$[u]_\alpha = 1, \tag{14}$$

$$[\beta u_x]_\alpha = 0, \tag{15}$$

$$u(x, 0) = 0, \quad x \in \Omega^- \cup \Omega^+. \tag{16}$$

We can write the general solution u of the NIPP in the form

$$u = u_0 + su_1 \tag{17}$$

for some $s \in \mathbb{R}$. It is easy to check that (1)-(2) and (4)-(5) hold. Furthermore, the parameter $s = [u]_\alpha$ is determined by the quadratic equation induced by the condition (3) (see Eq (106)). We shall call this method the $s-$parameter method. The idea of the method can be found more or less in p. 527 of [4] without justification. Note that it can also be potentially used when dealing with the counterpart elliptic model. However, some issues needed to be resolved before it can be justified and used.

a. Are the roots s all real?
b. If so, then we have two solutions and which one will lead to a physical (concentration) solution $0 < u < 1$?
c. Furthermore, can the method be used to the discretized version of NIPP as well?