INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 22, Number 6, Pages 843–859

doi: 10.4208/ijnam2025-1037

NUMERICAL ANALYSIS OF THE FINITE DIFFERENCE TIME DOMAIN METHODS WITH HIGH ACCURACY IN TIME FOR MAXWELL EQUATIONS

LIPING GAO, XIAOSONG ZHANG, AND RENGANG SHI*

Abstract. In this paper, we give a rigorous analysis of the finite difference time domain (FDTD) method with high accuracy in time (HAIT) (named HAIT-FDTD(M)) for the three dimensional Maxwell equations, where the time discretization is based on the Taylor expansion of the form: $U^n = C_n^0 + C_n^1\Delta t + \cdots + \frac{1}{M!}C_n^M(\Delta t)^M$ to approximate the fields in time. It is proven that the solutions of the schemes and the vectors representing the coefficients are divergence free. By using the energy method, the numerical energy identities of HAIT-FDTD(M) with $3 \leq M \leq 8$ are derived. It is then proved that these schemes are numerically and monotonically energy conserved as the polynomial degree M becomes large. With the help of the energy identities, stability conditions for the six schemes are derived, and how to select M and Δt in practice is given. By deriving error equations, we prove that the six schemes have convergence of the Mth order in time and the second order in space. Numerical experiments are provided and confirm the analysis on free divergence, approximate energy conservation, stability, and convergence.

Key words. Maxwell equations, finite difference time domain method, stability, energy conservation, convergence, Taylor expansion.

1. Introduction

The finite difference time domain (FDTD) method is one of the methods for numerical solutions of time dependent Maxwell equations, and causes many people's interests and much good research work. For example, the Yee scheme ([31]), proposed by Yee in 1966, is a very popular and efficient method (see Taflove [26]). Monk and S*ü*li [15] proved that the Yee scheme over non-uniform grids is of super convergence of second order in L^2 norm. For the Yee scheme in metamaterials, Li and Shields [12] proved that this scheme is also super convergent in L^2 norm. The stability and second order convergence of the Yee scheme under H^1 norm were proved in [7]. Recently, convergence analysis of the Yee schemes in linear dispersive media was given by Sakkaplankul and Bokil in [20]. The other FDTD methods, including the alternating direction implicit FDTD (ADI-FDTD) methods, the energy-conserved splitting FDTD methods, symplectic FDTD method, locally-one-dimensional (LOD) FDTD method, etc. and their analysis are seen in [33, 17, 6], [1, 2], [8, 10, 24, 25],[32], [9], [23], [3], [11], [4], [14, 29], [21], [30], [19, 27], [18] and the references therein.

Time discretization is important for accuracy, efficiency, stability, and convergence. There are many good time-stepping methods in numerical solutions of Maxwell equations [26, 16, 13]. For example, leap-frog method in [31, 18], Runge-Kutta method in [9, 21], ADI method in [33, 17], splitting methods in [11, 5], energy splitting conserving methods in [1, 2, 14, 30], symplectic method in [10, 25, 24], fourth order method based on the relation between time derivatives and spatial derivatives [29, 32], time-domain moment method based on weighted Laguerre

Received by the editors on December 31, 2024 and, accepted on March 14, 2025.
2000 *Mathematics Subject Classification.* 65M06, 65M12, 65M15, 78M20.
*Corresponding author.

polynomials in [3], Newmark time-stepping method in [4], LOD method in [23], Crank-Nicolson method in [28] and explicit-implicit hybrid time-stepping method in [19, 27] and the others in the references.

Different from the above FDTD methods, a new explicit FDTD method with high accuracy in time (HAIT)(called HAIT-FDTD(M)) was proposed in [22] by using Taylor expansion of the form: $U^n = C_n^0 + C_n^1\Delta t + \cdots + \frac{1}{M!}C_n^M(\Delta t)^M$ to approximate the fields in time, which transforms the Maxwell equations into a system of time-independent differential equations in the coefficients $C_n^k (k = 0, 1, \dots, M)$, and using the central difference methods to approximate the spatial derivatives of C_n^k. Numerical experiments demonstrate that HAIT-FDTD(M) has the following features: easy implementation, divergence free, numerical energy conservation, and good stability and convergence. However, the rigorous analysis of HAIT-FDTD(M) on stability, error estimate, and convergence by the energy method is not available, since the form of the scheme is very different from traditional ones, which makes the usual analysis methods on stability and convergence (see [15, 1, 7], etc.) do not work on HAIT-FDTD(M). In addition, how to select the polynomial degree M and time step sizes is not clear. Therefore, it is significant to give a rigorous analysis of HAIT-FDTD(M) on these issues.

In this paper, we analyze the HAIT-FDTD(M) schemes for the 3D Maxwell equations with perfectly electric conducting (PEC) boundary conditions. The research methods and results are as follows:

(i) It is proved that the solutions of the HAIT-FDTD(M) schemes and the representing coefficients retain the free divergence property.

(ii) By using the energy method, numerical energy identities of the HAIT-FDTD(M) schemes with $3 \le M \le 8$ are derived, and it is then proved that these schemes are approximately energy conserved. With the help of the energy identities, the stability conditions of the six schemes (which are weaker than the CFL (Courant-Friedrichs-Lewy) condition and can be used to select time step sizes and degree M) are derived, and the stability in L^2 norm is then proved.

(iii) It is proved that the HAIT-FDTD(M) schemes with $3 \le M \le 8$ are of convergence of M-th order in time and second order in space by using different error analysis from the traditional ones.

(iv) Numerical experiments are carried out and confirm the theoretical analysis of the schemes on free divergence, numerical energy conservation, good stability, and convergence.

2. Maxwell equations and some properties.

2.1. Maxwell equations and properties of the solution. Consider the 3D Maxwell equations in a domain of $\Omega \times (0, T]$:

$$\varepsilon\,\frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \qquad \mu\,\frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}, \tag{1}$$

$$\varepsilon\,\frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \qquad \mu\,\frac{\partial H_y}{\partial t} = \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}, \tag{2}$$

$$\varepsilon\,\frac{\partial E_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}, \qquad \mu\,\frac{\partial H_z}{\partial t} = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}, \tag{3}$$

where Ω is filled with homogeneous and isotropic medium, so the electric permittivity ε and the magnetic permeability μ are constants, and for $p = (x, y, z) \in \Omega$, $u = x, y, z$

$$E_u = E_u(p, t), \;\; H_u = H_u(p, t), \;\; (E_x, E_y, E_z) =: \mathbf{E}, \;\; (H_x, H_y, H_z) =: \mathbf{H}$$