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A MULTIGRID-BASED FOURTH ORDER FINITE DIFFERENCE METHOD FOR ELLIPTIC INTERFACE PROBLEMS WITH VARIABLE COEFFICIENTS

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Abstract. The paper introduces a fourth-order augmented matched interface and boundary (AMIB) method for solving elliptic interface problems with complex interfaces and piecewise smooth coefficients in two and three dimensions. To resolve the challenge posed by non-constant coefficients within the AMIB framework, the fast Fourier transform (FFT) Poisson solver of the existing AMIB methods is replaced by a geometric multigrid method to efficiently invert the Laplacian discretization matrix. In this work, a fourth order multigrid method will be employed in the framework of the AMIB method for elliptic interface problems with variable coefficients in two and three dimensions. Based on a Cartesian mesh, the standard fourth-order finite differences are employed to approximate the first and second derivatives involved in the Laplacian with variable coefficients. Near the interface, a fourth-order ray-casting matched interface and boundary (MIB) scheme is generalized to variable coefficient problems to enforce interface jump conditions in the corrected finite difference discretization. The augmented formulation of the AMIB allows us to decouple the interface treatments from the inversion of the Laplacian discretization matrix, so that one essentially solves an elliptic subproblem without interfaces. A fourth order geometric multigrid method is introduced to solve this subproblem with a Dirichlet boundary condition, where fourth order one-sided finite difference approximations are considered near the boundary in all grid levels. The proposed multigrid method significantly enhances the computational efficiency in solving variable coefficient problems, while achieving a fourth-order accuracy in accommodating complex interfaces and discontinuous solutions.

Key words. Variable coefficient elliptic interface problem, high order finite difference schemes, geometric multigrid, matched interface and boundary method (MIB), gradient recovery.

1. Introduction

This work focuses on solving multi-dimensional elliptic interface problems with variable coefficients. We consider an elliptic partial difference equation (PDE) in a domain Ω

$$\nabla \cdot (\beta \nabla u) + \kappa u = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1}$$

subject to Dirichlet boundary conditions on the boundary $\partial\Omega$. The function $u(\mathbf{x})$ together with the corresponding source term $f(\mathbf{x})$ depend on a vector variable $\mathbf{x} = (x_1, x_2, \cdots, x_d)$ for $d = 2$ or $d = 3$ on a rectangular or cubic domain Ω. The interface Γ defined by $\Gamma = \Omega^+ \cap \Omega^-$ divides the computational domain Ω into disjoint subdomains $\Omega = \Omega^+ \cup \Omega^-$. An illustration of subdomains in two dimensions is given in Fig. 1. The coefficients $\beta(\mathbf{x})$ and $\kappa(\mathbf{x})$ are smooth functions on each disjoint subdomain, but may be discontinuous across the interface Γ, i.e., they are piecewise smooth functions

$$\beta(\mathbf{x}) = \begin{cases} \beta^-(\mathbf{x}) & \text{in } \Omega^- \\ \beta^+(\mathbf{x}) & \text{in } \Omega^+, \end{cases} \quad \kappa(\mathbf{x}) = \begin{cases} \kappa^-(\mathbf{x}) & \text{in } \Omega^- \\ \kappa^+(\mathbf{x}) & \text{in } \Omega^+. \end{cases}$$

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Similarly, the source $f(\mathbf{x})$ is also piecewisely smooth with notation $f^+(\mathbf{x})$ and $f^-(\mathbf{x})$, respectively, in Ω^+ and Ω^-. It is assumed that $\beta(\mathbf{x})$ is always positive. Across the interface Γ, two jump conditions are known for the function and its flux in the normal direction

$$[\![u]\!] := u^+ - u^- = \phi(\mathbf{x}), \qquad \mathbf{x} \in \Gamma, \tag{2}$$

$$[\![\beta u_n]\!] := \beta^+(\mathbf{x})\nabla u^+ \cdot \vec{n} - \beta^-(\mathbf{x})\nabla u^- \cdot \vec{n} = \psi(\mathbf{x}), \qquad \mathbf{x} \in \Gamma, \tag{3}$$

where $\vec{n}$ is the outward normal direction of Γ pointing from Ω^- to Ω^+, and the superscript stands for the limiting value from each side of the interface. Equations (2) and (3) are called as the zeroth and first order jump conditions. Such an elliptic interface problem with discontinuous coefficients has wide application in a variety of fields such as fluid dynamics, modeling of underground waste disposal, solidification processes, oil reservoir simulations, and many others.

When the coefficients $\beta(\mathbf{x})$ and $\kappa(\mathbf{x})$ are piecewise constants, the present problem reduces to the usual elliptic interface problem, for which the finite element method (FEM) is a commonly used approach. Classical FEM [3, 7, 12, 35] delivers satisfactory accuracy, particularly when the interfaces align well with the underlying meshes. However, practical scenarios often necessitate the construction of numerical methods on non-fitted meshes. This requirement has driven the development of the Immersed FEM (IFEM) [31], in which local basis functions are adapted to ensure compliance with the prescribed jump conditions.

Finite difference methods on Cartesian grids have received extensive attention in the context of elliptic interface problems. Peskin [44] laid the foundation for this field by introducing a first-order accurate immersed boundary method in the 1970s. LeVeque and Li [34] pioneered the first second-order finite difference approach, the Immersed Interface Method (IIM), which employs Taylor series expansions to determine stencil weights. Another popular technique is the Ghost Fluid Method (GFM) [17], typically a first-order method [40], but it has been extended to second order in [39]. The recovery of flux convergence in GFM has been investigated in [16]. Chen et al. [11] developed a second-order compact finite difference method for solving elliptic interface problems. In addition to finite element and finite difference methods, other effective algorithms for solving elliptic interface problems include virtual node method [4, 28], finite volume method [6], and coupling interface method [13, 48]. We note that the aforementioned methods usually deliver first or second order accuracy.

Addressing variable coefficients in elliptic interface problems is a challenging endeavor. These variable coefficients could exhibit significant variations across the interface, causing abrupt shifts in solutions. Managing such discontinuities presents a huge difficulty in maintaining numerical stability and precision. Due to the inherent complexity, only a limited number of studies have ventured into this domain. A weak formulation has been developed in [30] for a grid that fits the geometry of the problem, offering a solution to variable coefficient elliptic equations. Remarkably, it only requires Lipschitz continuity rather than smoothness for the interfaces. Expanding upon the foundational principles of the IIM, novel techniques like the Decomposed Immersed Interface Method (DIIM) [5] and Augmented Immersed Interface Method (AIIM) [37] have been proposed to address elliptic interface problems with variable coefficients. A second-order finite-volume method is presented in [42] that operates on Cartesian grids for variable coefficient elliptic equations with embedded interfaces. Ref. [41] describes a composite spectral scheme for solving