INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 1, Pages 1–18

A POSTERIORI ERROR ESTIMATION FOR NON-CONFORMING QUADRILATERAL FINITE ELEMENTS

MARK AINSWORTH

(Communicated by Zhimin Zhang)

Abstract. We derive an a posteriori error estimator giving a computable upper bound on the error in the energy norm for finite element approximation using the non-conforming rotated $\mathbb{Q}_1$ finite element. It is shown that the estimator also gives a local lower bound up to a generic constant. The bounds do not require additional assumptions on the regularity of the true solution of the underlying elliptic problem and, the mesh is only required to be locally quasi-uniform and may consist of general, non-affine convex quadrilateral elements.

Key Words. A posteriori error estimation, non-conforming finite elements, rotated $\mathbb{Q}_1$ element, non-affine quadrilateral elements.

1. Introduction

Non-conforming finite element methods are of considerable interest in the numerical approximation of elliptic partial differential equations where issues of stability and locking [4] may render a conforming scheme practically useless. A large number of non-conforming finite element methods [7] were developed in the engineering community on a more or less *ad hoc* basis and found to produce excellent numerical results in practice. The mathematical support for such elements only came at a later stage, followed by the development of new non-conforming elements accompanied by proofs of stability and convergence [8, 14].

Whilst the topic of a posteriori error estimation for *conforming* finite element methods has now matured to a high level of sophistication [2, 3, 15], the situation regarding *non-conforming* finite element schemes is at a relatively primitive stage. An early important contribution to the theory of a posteriori error estimation for the non-conforming $\mathbb{P}_1$ triangular finite element of Crouzeix-Raviart [8] was made by Dari et. al. [10] who obtained two sided bounds on the error measured in the energy norm up to generic constants. These ideas were later extended to non-conforming mixed finite element approximation of Stokes flow [9] using the Crouzeix-Raviart finite element. Hierarchical basis type estimators were explored in [11], whilst [5] derived estimators based on gradient averaging techniques. More recently, a new a posteriori error estimator was derived [1] and shown to provide two-sided bounds on the error and, significantly, the upper bound does not involve *any* generic constants meaning that one has a guaranteed computable upper bound on the error measured in the energy norm.

Received by the editors October 6, 2004 and, in revised form, November 5, 2004.
2000 *Mathematics Subject Classification.* 65N50. Secondary 65N15, 65N50, 76S05.
Support for this work from the Leverhulme Trust through a Leverhulme Trust Fellowship is gratefully acknowledged.

The aim of the present work is to extend the ideas of [1] to the non-conforming *rotated* $\mathbb{Q}_1$ element of Rannacher and Turek [14] for meshes of quadrilaterals. The study of approximation properties on quadrilateral elements is rather delicate owing to the fact that the mapping from the standard reference element to the physical element is in general non-affine. This is exacerbated by the fact that, despite its name, the rotated $\mathbb{Q}_1$ element does not contain the full approximation space $\mathbb{Q}_1$, even in the case of affine elements. Together, these effects may even lead to non-convergence of the approximation error under certain unfavourable circumstances. Nevertheless, conditions on the mesh under which the element is able to produce an optimal rate of convergence are well-understood [12, 13] in the context of a priori error estimation where, roughly speaking, it is found that the elements should not be too distorted from parallelograms.

Here, we derive a computable a posteriori error estimator that produces an upper bound on the error in the energy norm that is valid even for non-affine elements. Moreover, it is shown that the estimator is efficient in the sense that it also gives a lower bound up to a generic constant independent of the mesh-size. The bounds are obtained without making any additional assumptions on the regularity of the true solution of the underlying elliptic problem, and the mesh is only required to be locally quasi-uniform, thereby allowing the use of an adaptive local refinement algorithm.

In view of the difficulties in the a priori convergence estimates, one might suspect the upper bound property of the a posteriori error estimator to degenerate on meshes containing elements that are too highly distorted. This proves not to be the case, and it is worth emphasising that our upper bound remains valid under the very mild assumption that the elements are convex. Of course, the effects of element distortion may well mean that, by analogy with the actual error, the estimator converges at a sub-optimal rate. This is to be expected from a reliable and efficient estimator, but it should be borne in mind that this is a defect of the underlying mesh and approximation scheme and not of the a posteriori error estimator. On the contrary, the availability of a computable upper bound means that one can actually use elements that are *more distorted* than one might have been comfortable with from the a priori viewpoint, secure in the knowledge that if the estimator is sufficiently small, then the overall approximation is acceptable thanks to the upper bound property of the estimator.

2. Model Problem and Its Non-conforming Discretisation

Consider the model problem of finding u such that

$$-\operatorname{\mathbf{div}}(a\,\mathbf{grad}\,u) = f \quad \text{in } \Omega \tag{1}$$

subject to $u = 0$ on Γ_D and $\boldsymbol{n} \cdot a\,\mathbf{grad}\,u = g$ on Γ_N, where Ω is a planar polygonal domain and the disjoint sets Γ_D and Γ_N form a partitioning of the boundary of Ω. The data satisfy $f \in L_2(\Omega)$, $g \in L_2(\Gamma_N)$ and $a \in L_\infty(\Omega)$ is assumed non-negative. For simplicity, we shall assume that a is piecewise constant on the finite element mesh.

The variational form of the problem consists of seeking $u \in H^1_E(\Omega)$ such that

$$(a\,\mathbf{grad}\,u, \mathbf{grad}\,v) = (f, v) + \int_{\Gamma_N} gv\,\mathrm{d}s \quad \forall v \in H^1_E(\Omega) \tag{2}$$

where $H^1_E(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$. The notation $(\cdot,\cdot)_\omega$ is used to denote the L_2-inner product over a domain ω, with the subscript omitted where ω is the physical domain Ω. The corresponding norm is denoted by $\|\cdot\|_\omega$.