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A New 6-point Ternary Interpolating Subdivision Scheme and its Differentiability

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Abstract. We present a new 6-point ternary interpolating scheme with a shape parameter. The scheme is C^2 continuous over the parametric interval. The differentiable properties of proposed as well as two other existing 6-point ternary interpolating schemes have been explored. Application of proposed scheme is given to show its visual smoothness.

Keywords: Interpolating subdivision scheme, continuity, smoothness, shape parameter, Laurent polynomial.

1. Introduction

Computer Aided Geometric Design (CAGD) is a branch of applied mathematics, which deals with algorithms for the shape and structure of smooth curves and surfaces and for their competent mathematical demonstration. Subdivision is a very common approach which is related to CAGD. We can survey subdivision as process of taking a coarse shape and refining it to produce another shape that is more visually nice-looking and smooth. We can divide the subdivision schemes into two major types: one is interpolating, in which original points stay undistributed while new points are included and the other is approximating, in which new points are included as well as old points are moved at each refinement level.

Now a days wide variety of 6-point interpolating binary/ternary schemes have been introduced in the literature. Deslauries and Dubuc [1] introduced 6-point ternary interpolating scheme in 1989. In [7] Weisman described a 6-point binary interpolating scheme. Khan and Mustafa [6] introduced ternary 6-point subdivision scheme. Lian [5] generalized classical 4-point and 6-point interpolating schemes to a-ary interpolating schemes for any integer $a \geq 3$. The Laurents polynomial method has been used by [2] to discuss analysis of binary/ternary schemes.

A general ternary subdivision scheme S which maps a coarse polygon $f^k = \{f_i^k\}_{i\in Z}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in Z}$ is defined by

$$f_i^{k+1} = \sum_{j\in Z} a_{3j-i} f_j^k, \qquad i \in Z, \tag{1.1}$$

where the set $a = \{a_i \mid i \in Z\}$ of coefficients is called mask of the scheme. A necessary condition for uniform convergence of the subdivision scheme (1.1) is that

$$\sum_{j\in Z} a_{3j} = \sum_{j\in Z} a_{3j+1} = \sum_{j\in Z} a_{3j+2} = 1 \tag{1.2}$$

The z-transform of the mask a of subdivision scheme can be given as

$$a(z) = \sum_{i\in Z} a_i z^i, \tag{1.3}$$

which is called the symbol or Laurent polynomial of the scheme. From (1.2) and (1.3) the Laurent polynomial of a convergent subdivision scheme satisfies

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$$a\left(e^{2i\pi/3}\right) = a\left(e^{4i\pi/3}\right) = 0 \quad \text{and} \quad a(1) = 3. \tag{1.4}$$

The existence of associated subdivision scheme for the divided differences of the original control polygon and of related Laurent polynomial $a_1(z)$ is assured by this condition

$$a_1(z) = \frac{3z^2}{z^2 + z + 1} a(z).$$

The subdivision scheme S_1 with symbol $a_1(z)$ is connected to scheme S with symbol $a(z)$ by the following theorem.

Theorem 1.1. [3] Let a subdivision scheme is denoted by S with symbol $a(z)$ satisfying (1.4). Then there exist a subdivision scheme S_1 with the property

$$\Delta f^k = S_1 \Delta f^{k-1},$$

where $f^k = S^k f^0$ and $\Delta f^k = \left\{(\Delta f^k)_i = 3^k\left(f_{i+1}^k - f_i^k\right) : i \in Z\right\}$. Moreover, S is uniformly convergent if and only if $\frac{1}{3}S_1$ converges uniformly to the zero function for all initial data f^0, such that

$$\lim_{k\to\infty}\left(\frac{1}{3}S_1\right)^k f^0 = 0.$$

We define the norm of scheme as

$$\left\|\left(\frac{1}{3}S_n\right)^L\right\|_\infty = \max\left\{\sum_{j\in Z}\left|b_{i+3^L j}^{[n,L]}\right| : i = 0,1, \ldots, 3^L - 1\right\}, \tag{1.5}$$

where

$$b^{[n,L]}(z) = \tfrac{1}{3^L}\prod_{j=0}^{L-1} a_n\left(z^{3^j}\right). \tag{1.6}$$

and

$$a_n(z) = \left(\frac{3z^2}{z^2+z+1}\right) a_{n-1}(z) = \left(\frac{3z^2}{z^2+z+1}\right)^n a(z), \quad n \geq 1. \tag{1.7}$$

Theorem 1.2. [3] Let S be subdivision scheme with a characteristic II-polynomial $a(z) = \left(\frac{z^2+z+1}{3z^2}\right)^n q(z), q \in \mathrm{II}$. If the subdivision scheme S_n, corresponding to the II-polynomial $q(z)$, converges uniformly then $S^\infty f^0 \in C^n(R)$ for any initial control polygon f^0.

Corollary 1.3. [3] If S is a subdivision scheme of the form above and $\frac{1}{3}S_{n+1}$ converges uniformly to the zero function for all initial data f^0 then $S^\infty f^0 \in C^n(R)$ for any initial control polygon f^0

2. A 6-point ternary interpolating scheme

In this section, we construct a 6-point ternary interpolating subdivision scheme.

2.1. Construction of the scheme

Consider the following three recursive relations which refine given *k*th level polygon $f^k = \{f_i^k\}_{i\in Z}$ to (*k*+1)th level polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in Z}$

$$\begin{aligned}
f_{3i}^{k+1} &= f_i^k, \\
f_{3i+1}^{k+1} &= a_0 f_{i-2}^k + a_1 f_{i-1}^k + a_2 f_i^k + a_3 f_{i+1}^k + a_4 f_{i+2}^k + a_5 f_{i+3}^k, \\
f_{3i+2}^{k+1} &= a_5 f_{i-2}^k + a_4 f_{i-1}^k + a_3 f_i^k + a_2 f_{i+1}^k + a_1 f_{i+2}^k + a_0 f_{i+3}^k.
\end{aligned} \tag{2.1}$$

We get following mask from above recursive relations