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An Improvement on Disc Separation of the Schur Complement and Bounds for Determinants of Diagonally Dominant Matrices

Zhuohong Huang, Tingzhu Huang +

School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 610054, P. R. China

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Abstract. In this paper, we improve the disc separation of the Schur complement of strictly diagonally dominant matrices presented in Liu [SIAM. J. Matrix Anal. Appl., 27 (2005): 665-674]. As applications, we present some new bounds for determinants of original matrices and estimations for eigenvalues of Schur complement. By theoretical analysis, we improve the bounds of determinants established in Huang [*Comput. Math. Appl.*, 50 (2005): 1677-1684].

Keywords: *H*-matrix; strictly (doubly) diagonally dominant matrix; Schur complement; Geršgorin's theorem.

1. Introduction

For localization of eigenvalues and estimations of determinants, many researches have been proposed, e.g., [1-5]. Recently Liu [6] discussed the diagonally dominant degree of the Schur complement of strictly diagonally dominant matrices and presented the localization for eigenvalues of the Schur complement and some bounds for determinants of the strictly diagonally dominant matrices. Huang [7] estimated the bounds for determinants of diagonally dominant matrices, general H-matrices and certain not diagonally dominant matrices. In this paper, we improve the diagonally dominant degree of the Schur complement of diagonally dominant matrices in [6]. Further, we obtain new bounds for determinants of diagonally dominant matrices and the estimations of eigenvalues of the Schur complement, these results improve the estimations of [6,7].

Let $A \in C^{n\times n}$ be a strictly diagonally (row) dominant matrix (SD_n), if and only if

$$|a_{ii}| > P_i(A),\ \ P_i(A) = \sum_{j=1,\, j\neq i}^{n} |a_{ij}|\,(\text{abbreviated } P_i),\ \ \forall i = 1,2,\cdots,n. \tag{1}$$

Let $A \in C^{n\times n}$ be a strictly doubly diagonally (row) dominant matrix (SDD_n), if and only if

$$|a_{ii}||a_{jj}| > P_i(A)P_j(A), \forall i, j = 1,2,\cdots,n. \tag{2}$$

If $A \in \mathrm{SDD}_n$, but $A \notin \mathrm{SD}_n$, then, by (2), there exists a unique i_0 such that

$$|a_{i_0,i_0}| \le P_{i_0}(A). \tag{3}$$

For $A = (a_{ij})$ and $B = (b_{ij}) \in C^{m\times n}$, we write $A \ge B$, if $a_{ij} \ge b_{ij}$ for all i, j. A real $n\times n$ matrix A is called an M-matrix (M_n) if $A = sI_n - B$, where $s \ge 0, B \ge 0$ and $s > \rho(B)$, $\rho(B)$ is the spectral radius of B.

Suppose $A \in C^{n\times n}$, A be called an H-matrix (H_n) if $\mu(A) \in M_n$, where, the comparison matrix $\mu(A) = (\mu_{ij})$ be defined by

+ Corresponding author. Tel.: 15928746831.
E-mail address: zhuohonghuang@yahoo.cn.

$$\mu_{ij} = \begin{cases} -|a_{ij}|, i \neq j, \\ |a_{ij}|, i = j, \end{cases} \quad i, j = 1, 2, \cdots, n.$$

Let x^T denote the transpose of the vector x, and I_n denote the $n \times n$ identity matrix. Let $A \in C^{n \times n}$, and $N = \{1, 2, \cdots, n\}$. If $\alpha \subseteq N$, $|\alpha|$ equals the cardinality of α. For nonempty index sets $\alpha, \beta \subseteq N$, we denote by $A(\alpha, \beta)$ the submatrix of A lying in the rows indicated by α and the columns indicated by β. The submatrix $A(\alpha, \alpha)$ be abbreviated to $A(\alpha)$. Let $\alpha \subset N$ and $\alpha^c = N - \alpha$, both arranged in increasing order. Then

$$A/\alpha = A/A(\alpha) = A(\alpha^c) - A(\alpha^c, \alpha)[A(\alpha)]^{-1} A(\alpha, \alpha^c)$$

be called the Schur complement with respect to $A(\alpha)$.

Lemma 1.1 (See [8]). Let $A \in M_n$. Then there exists a positive diagonal matrix D such that $AD \in \mathrm{SD}_n$.

Lemma 1.2. (See [12]). Let $A \in \mathrm{SD}_n, \mathrm{SDD}_n$. Then $\mu(A) \in M_n$, $A \in H_n$.

Lemma 1.3 (See [9]). Let $A \in C^{n \times n}$, $B \in M_n$. If $\mu(A) \geq B$, then $A \in H_n$ and $B^{-1} \geq |A^{-1}| \geq 0$.

Remark 1.1. From Lemma 1.3, we obtain immediately that

$$A \in H_n \Rightarrow [\mu(A)]^{-1} \geq |A^{-1}|.$$

Lemma 1.4 (See [10]). Let $A \in SD_n$ and m be a proper subset of n. Then

$$A/m \in \mathrm{SD}_{n-|m|}.$$

Lemma 1.5 (See [11]). Let $A \in C^{n \times n}$. A is an H matrix if the following inequality be hold

$$|a_{ii}| > \sum_{\substack{t \in N_1 \\ t \neq i}} |a_{it}| + \mu \sum_{t \in N_2} |a_{it}| \frac{P_t}{|a_{tt}|}, \quad \forall i \in N_1, \tag{4}$$

where

$$0 \leq \mu \triangleq \begin{cases} \max\limits_{1 \leq \omega \leq k} \dfrac{\sum\limits_{t \in N_1} |a_{jt}|}{P_j - \sum\limits_{t \in N_2, t \neq j} |a_{jt}| \frac{P_t}{|a_{tt}|}} \leq 1, & \text{if} \quad P_t \neq 0, \forall j \in N_2, \\ 1 & \text{else if } \exists j \in N_2, \hat{P}_j = 0, \end{cases}$$

$$\hat{P}_t \triangleq \sum_{\substack{t \in N_1 \\ t \neq i}} |a_{it}| + \sum_{t \in N_2} |a_{it}| \frac{P_t}{|a_{tt}|}, \quad \forall i \in N_1,$$

$$N_1 \triangleq \{i \in N \mid 0 < |a_{ii}| \leq P_i(A)\}, N_2 \triangleq \{i \in N \mid |a_{ii}| > P_i(A)\}.$$

2. Disc separation of the Schur complements of SD_n and SDD_n

In this section, by discussing the criteria of H_n, we improve the diagonally dominant degree of the Schur complement of SD_n and SDD_n in [6].

Lemma 2.1. Let $A \in \mathrm{SD}_n(\text{or } \mathrm{SDD}_n)$, $\alpha = \{i_1, i_2, \cdots, i_k\}$ be a proper subset of N and $\alpha^c = N - \alpha$ $= \{j_1, j_2, \cdots, j_l\}$, $k + l = n$. For any $j_t \in \alpha^c$, denote