ISSN 1746-7659, England, UK
Journal of Information and Computing Science
Vol. 5, No. 3, 2010, *pp*. 233-240

Sign Idempotent Matrices and Generalized Inverses

Jingyue Zhang [1,+], T. Z. Huang [1], Zhongshan Li [2] and Xingwen Zhu [1]

[1]School of Mathematical Sciences, University of Electronic Science and Technology, 610054

[2]Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30302-4110, USA

(*Received December 14, 2009, accepted December 22, 2009*)

Abstract. A matrix whose entries consist of $\{+,-,0\}$ is called a sign pattern matrix. Let $Q(A)$ denote the set of all real $n\times n$ matrices B such that the signs of entries in B match the corresponding entries in A. For nonnegative sign patterns, sign idempotent patterns have been characterized. In this paper, we Firstly give an equi-valent proposition to characterize general sign idempotent matrices (sign idempotent). Next, we study properties of a class of matrices which can be generalized permutationally similar to specialized sign patterns. Finally, we consider the relationships among the allowance of idempotent, generalized inverses and the allowance of tripotent in symmetric sign idempotent patterns.

Keywords: sign pattern matrix; symmetric sign pattern; sign idempotent; allowance of idempotent; generalized inverses

1. Introduction

A matrix whose entries consist of $\{+,-,0\}$ is called a sign pattern matrix. A subpattern of a sign pattern ma-trix A is a sign pattern matrix obtained by replacing some of the $+$ or $-$ entries in A with 0. The sign pattern I_n is the diagonal pattern of order n with + diagonal entries. In order to study conveniently, we also ues I to denote diagonal patterns with + diagonal entries. A sign pattern is said to be sign nonsingular, if all the real matrices $B\in Q(A)$ are nonsingular. A is said to be sign singular if every matrix $B\in Q(A)$ is a singular matrix.

A permutation pattern is a square sign pattern with entries 0 and +, where the entry + occurs precisely once in each row and column. A permutational similarity of the square pattern A is a product of the form $P^T AP$, where P is a permutation pattern. A signature pattern is a diagonal sign pattern matrix each of whose diagonal entries is + or -. A generalized permutation pattern is either a permutation pattern or a signature pattern obtained by replacing some or all of the + entries in a permutation pattern with – entries. A is a symmetric sign pattern matrix if the entries of A satisfy $a_{i,j}=+(-or0)$ if and only if $a_{i,j}=+(-or0)$ for any i and j.

A matrix A is called constantly signed if it is of the form $A=\partial J$, where $\partial\in(+,-,0)$ and J is a sign pattern matrix whose entries are all positive. If A is said to be row (column) constantly signed if entres in rows (columns) have the same sign. For a sign pattern matrix A, we can define A^2 as a sign pattern matrix if no two nonzero terms in the sum $\sum_k a_{i,k}a_{k,j}$ are oppositely signed for all i and j; otherwise A^2 is not a sign pattern. if $A^2=A$, then A is called sign idempotent. A sign pattern matrix A is said to be allowed idempotent (the allow-ance of idempotent) if there exists $B\in Q(A)$, where $B^2=B$. Since an irreducible sign idempotent matrix has been characterized, for a reducible sign pattern matrix, we often assume a sign pattern matrix A is in Frobenius normal form, i.e.

+ Corresponding author. *E-mail address*: jyzhang2046@163.com

$$A=\begin{pmatrix} A_{11} & \cdots & A_{1k} \\ & \ddots & \vdots \\ & & A_{kk} \end{pmatrix} \tag{1}$$

where each A_{ii} is square and irreducible or A_{ii} is a 0-entry, denoted by (0).

In [1], Eschenbach defined the modified Frobenius normal form, which is a sign pattern matrix A whose form is as follow:

$$A=\begin{pmatrix} A_{11} & \cdots & A_{1k} \\ & \ddots & A_{kk} \end{pmatrix} \tag{2}$$

where each A_{ii} is either positive or entry-wise zero.

For a sign pattern matrix A, the minimum rank of A denoted by $mr(A)$ is defined as

$$mr(A)=\min\{rank(B)|B\in Q(A)\}.$$

Now, we will give several notations about generalized inverses to denote the class of sign pattern matrices.

1. A sign pattern matrix A is said to allow idempotent (the allowance of idempotent) if there exists $B\in Q(A)$, where $B^2=B$. We denote ID as the class of all square sign patterns which have the property.

2. A sign pattern matrix A is said to allow tripotent (the allowance of tripotent) if there exists $B\in Q(A)$, where $B^3=B$. We denote T as the class of all square sign patterns which have the property.

3. A sign pattern matrix A is said to allow (1)-inverse if there exists $B,C\in Q(A)$, where $BCB=B$. We denote G as the class of all square sign patterns which have the property.

Let F be the class of all square sign pattern matrices A such that A are generalized permutationally similar to a matrix of the form

$$F\left(I_r,A_2,A_3\right)=\begin{pmatrix} I_r & A_2 \\ A_3 & A_3A_2 \end{pmatrix}, \tag{3}$$

where A_2A_3 is a subpattern of I_r.

In this paper, we investigate general sign idempotent matrices and generalized inverses. In [3], for non-negative sign patterns, sign idempotent patterns have been characterized. In section 2, we give an equivalent proposition to characterize general sign idempotent patterns. Moreover, we will demonstrate that any symme-tric idempotent sign pattern matrix has the form of $\begin{pmatrix} I_r & A_2 \\ A_2^T & A_2^TA_2 \end{pmatrix}$, where $A_2A_2^T$ is a subpattern of I_r. In [2], for nonnegative sign patterns, which are permutationally similar to $\begin{pmatrix} T & A_2 \\ A_3 & A_3TA_2 \end{pmatrix}$, where A_2A_3 is diagonal, and $T^2=I$, have have been characterized. In section 3, we extend its properties, several properties of F are characterized. Finally, some equivalent properties are given in symmetric sign idempotent patterns.

2. An equivalent form of sign idempotent matrices

From [3], we know any nonnegative sign idempotent pattern matrix can be permutationally similar to a block matrix. In this section, we generalize Theorem 3.2 of [3] to general sign patterns. Furthermore, we obtain an equivalent proposition for general sign idempotent matrices.

Lemma 1. [2] Each of the classes T, ID and G is closed under the following operations:

1. signature similarity;
2. permutation similarity;
3. transposition.