

(2,1)-Total Labelling of Cactus Graphs

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Abstract. A (2,1)-total labelling of a graph $G = (V, E)$ is an assignment of integers to each vertex and edge such that: (i) any two adjacent vertices of G receive distinct integers, (ii) any two adjacent edges of G receive distinct integers, and (iii) a vertex and its incident edge receive integers that differ by at least 2. The *span* of a (2,1)-total labelling is the maximum difference between two labels. The minimum span of a (2,1)-total labelling of G is called the (2,1)-total number and denoted by $\lambda_2^t(G)$.

A cactus graph is a connected graph in which every block is either an edge or a cycle. In this paper, we label the vertices and edges of a cactus graph by (2,1)-total labelling and have shown that, $\Delta + 1 \leq \lambda_2^t(G) \leq \Delta + 2$ for a cactus graph, where Δ is the degree of the graph G .

Keywords: Graph labelling; (2,1)-total labelling; cactus graph

1. Introduction

Motivated by frequency channel assignment problem Griggs and Yeh [5] introduced the $L(2,1)$ -labelling of graphs. The notation was subsequently generalized to the $L(p, q)$ -labelling problem of graphs. Let p and q be two non-negative integers. An $L(p, q)$ -labelling of a graph G is a function c from its vertex set $V(G)$ to the set $\{0, 1, \dots, k\}$ such that $|c(x) - c(y)| \geq p$ if x and y are adjacent and $|c(x) - c(y)| \geq q$ if x and y are at distance 2. The $L(p, q)$ -labelling number $\lambda_{p,q}(G)$ of G is the smallest k such that G has an $L(p, q)$ -labelling c with $\max\{c(v) \mid v \in V(G)\} = k$.

The $L(p, q)$ -labelling of graphs has been studied rather extensively in recent years [2, 8, 12, 16, 17, 18].

Whittlesey et al. [19] investigated the $L(2,1)$ -labelling of incidence graphs. The incidence graph of a graph G is the graph obtained from G by replacing each edge by a path of length 2. The $L(2,1)$ -labelling of the incidence graph G is equivalent to each element of $V(G) \cup E(G)$ such that:

- (i) any two adjacent vertices of G receive distinct integers,
- (ii) any two adjacent edges of G receive distinct integers, and
- (iii) a vertex and an edge incident receive integers that differ by at least 2.

This labelling is called (2,1)-total labelling of graphs which introduced by Havet and Yu [6] and generalized to the $(d,1)$ -total labelling, where $d \geq 1$ be an integer. A k -($d,1$)-total labelling of a graph G is a function c from $V(G) \cup E(G)$ to the set $\{0, 1, \dots, k\}$ such that $c(u) \neq c(v)$ if u and v are adjacent and $|c(u) - c(e)| \geq d$ if a vertex u is incident to an edge e . The $(d,1)$ -total number, denoted by $\lambda_d^t(G)$, is the least integer k such that G has a k -($d,1$)-total labelling. When $d = 1$, the (1,1)-total labelling is well known as total colouring of graphs.

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Let $\Delta(G)$ (or simply Δ) denote the maximum degree of a graph G .

Havet and Yu [6] proposed the following conjecture.

Conjecture 1 $\lambda_d^t(G) \leq \min\{\Delta + 2d - 1, 2\Delta + d - 1\}$.

2. Some general bounds of $(d,1)$ -total labelling

It is shown in [6] that for any graph G ,

- (i) $\lambda_d^t(G) \leq 2\Delta + d - 1$;
- (ii) $\lambda_d^t(G) \leq 2\Delta - 2\log(\Delta + 2) + 2\log(16d - 8) + d - 1$; and
- (iii) $\lambda_d^t(G) \leq 2\Delta - 1$ if $\Delta \geq 5$ is odd.

Again in [6] it was shown that

- (i) $\lambda_d^t(G) \geq \Delta + d - 1$;
- (ii) $\lambda_d^t(G) \geq \Delta + d$ if G is Δ -regular;
- (iii) $\lambda_d^t(G) \geq \Delta + d$ if $d \geq \Delta$; and
- (iv) $\lambda_d^t(G) \leq \chi(G) + \chi'(G) + d - 2$, where $\chi(G)$ and $\chi'(G)$ are known as chromatic number and chromatic index of G respectively.

Let $\text{Mad}(G)$ is the maximum average degree of G , $\text{Mad}(G) = \max\{2|E(H)|/|V(G)|, H \subseteq G\}$. Montassier and Raspaud [15] proved that if G be a connected graph with maximum degree Δ , $d \geq 2$, then $\lambda_d^t(G) \geq \Delta - 2d - 2$ in the following cases:

- (i) $\Delta \geq 2d + 1$ and $\text{Mad}(G) < \frac{5}{2}$;
- (ii) $\Delta \geq 2d + 2$ and $\text{Mad}(G) < 3$;
- (iii) $\Delta \geq 2d + 3$ and $\text{Mad}(G) < \frac{10}{3}$.

For a complete graph K_n , the result for $(d,1)$ -total labelling is given in [6]. If n is odd then $\lambda_2^t(K_n) = \min\{n + 2d - 2, 2n + d - 2\}$; if n is even then $\lambda_2^t(K_n) = \min\{n + 2d - 2, 2n + d - 2\}$, $n \leq d + 5$, $\lambda_2^t(K_n) = n + 2d - 1$, $n > 6d^2 - 10d + 4$ and $\lambda_2^t(K_n) \in \{n + 2d - 2, 2n + d - 1\}$ otherwise. Then they focused in $(2,1)$ -total labelling and shown that if $\Delta \geq 2$, then $\lambda_2^t(K_n) \leq 2\Delta + 2$ and therefore the $(d,1)$ -total labelling conjecture is true when $p = 2$ and $\Delta = 3$. In fact, the bound for this special case is tight as $\lambda_2^t(K_4) = 7$ [6].

In [13], Molloy and Reed proved that the total chromatic number of any graph with maximum degree Δ is at most Δ plus an absolute constant. Moreover, in [14], they gave a similar proof of this result for sparse graphs.

In [7], it was shown that for any tree T , $\Delta + 1 \leq \lambda_2^t(T) \leq \Delta + 2$, where Δ is the maximum degree among all the vertices of the tree.

The $(d,1)$ -total labelling for a few special graphs have been studied in literature, e.g., complete graphs [6], complete bipartite graphs [11], planar graphs [1], outer planar graphs [3], products of graphs [4], graphs with a given maximum average degree [15], etc. A more generalization of total colouring of graphs so called $[r, s, t]$ -colouring, was defined and investigated in [9].

It is shown in [10] that for any cactus graphs, $\Delta + 1 \leq \lambda_{2,1} \leq \Delta + 3$. Now in this paper, we label the vertices and edges of a cactus graphs G by $(2,1)$ -total labelling and it is shown that $\Delta + 1 \leq \lambda_2^t \leq \Delta + 2$.