

Perron Complements of Strictly Generalized Doubly Diagonally Dominant Matrices

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Abstract. As is known, Meyer introduced the concept of the Perron complements of nonnegative irreducible matrices. In addition, the Schur complements of generalized doubly diagonally dominant matrices were introduced by Liu et al. [Linear Algebra Appl., 378(2004): 231-244]. In this paper, properties of the Perron complement of strictly generalized doubly diagonally dominate matrices are presented.

Keywords: generalized doubly diagonally dominant matrix; diagonally dominant matrix; nonnegative irreducible matrix; Perron complement; Schur complement

1. Introduction

Let $A = (a_{ij})$ be a $n \times n$ matrix, and recall that A is (row) diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, i = 1, 2, \dots, n. \quad (1)$$

A is further said to be strictly diagonally dominant if all the inequalities of (1) are strict. A doubly diagonally dominant matrix (see, e.g., [6]) is a matrix such that for all $i \neq j$

$$|a_{ii}| |a_{jj}| \geq \sum_{\substack{t=1 \\ t \neq i}}^n |a_{it}| \sum_{\substack{t=1 \\ t \neq j}}^n |a_{jt}|, \quad (2)$$

and that A is strictly doubly diagonally dominant if all the inequalities of (2) are strict. We call A a generalized doubly diagonally dominant matrix (see, e.g., [6]) if there exist proper subset γ_1, γ_2 of $\langle n \rangle := \{1, 2, \dots, n\}$ such that $\gamma_1 \cap \gamma_2 = \emptyset$, $\gamma_1 \cup \gamma_2 = \langle n \rangle$ and

$$(|a_{ii}| - \alpha_i)(|a_{jj}| - \beta_j) \geq \beta_i \alpha_j. \quad (3)$$

for all $i \in \gamma_1, j \in \gamma_2$ where

$$\alpha_i = \sum_{\substack{t \neq i \\ t \in \gamma_1}} |a_{it}|, \quad \beta_i = \sum_{\substack{t \neq i \\ t \in \gamma_2}} |a_{it}|.$$

We call A a strictly generalized doubly diagonally dominant matrix if all the inequalities of (3) are strict.

Assuming that the matrix order is $n \geq 2$, we use the same notation as in paper [6]: D_n for diagonally dominant matrices; SD_n for strictly diagonally dominant matrices; DD_n for doubly diagonally dominant matrices; SDD_n for strictly doubly diagonally dominant matrices; $GDD_n^{\gamma_1, \gamma_2}$ for generalized doubly diagonally dominant matrices; $SGDD_n^{\gamma_1, \gamma_2}$ for strictly generalized doubly diagonally dominant.

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Let $A \in Z \equiv \{(a_{ij}) \in R^{n,n} : a_{ij} \leq 0, i \neq j\}$. If

$$A = aI - B, B \geq 0, a > \rho(B),$$

then A is called an M-matrix. The absolute matrix of A is defined by $|A| = (|a_{ij}|)$. The comparison matrix $\mu(A) = (\mu_{ij})$ is defined by

$$\mu_{ij} = \begin{cases} -|a_{ij}|, & i \neq j, \\ |a_{ij}|, & i = j. \end{cases}$$

A is called an H-matrix if $\mu(A)$ is an M-matrix. In the following we denote M-matrices and H-matrices by M_n and H_n , respectively.

Let α, β be nonempty ordered subsets of $\langle n \rangle$, both consisting of strictly increasing integers. By $A(\alpha, \beta)$ we shall denote the submatrix of A lying in rows indexed by α and columns indexed by β . If, in addition, $\alpha = \beta$, then the principal submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$.

Suppose that $\alpha \subset \langle n \rangle$. If $A(\alpha)$ is nonsingular, then the Schur complement of $A(\alpha)$ in A is given by

$$S(A/A(\alpha)) = A(\beta) - A(\beta, \alpha)[A(\alpha)]^{-1}A(\alpha, \beta), \quad (4)$$

where $\beta = \langle n \rangle \setminus \alpha$. Furthermore, the Schur complements have been well studied for various classes of matrices, including: positive definite matrices, M-matrices, inverse M-matrices and totally nonnegative matrices. A well-known result due to Carlson and Markham [1] states that the Schur complements of strictly diagonally dominant matrices are diagonally dominant. And the Schur complement of a generalized doubly diagonally dominant matrix is a generalized doubly diagonally dominant matrix (see, [6]).

A remarkable Schur formula is ([8])

$$\det(S(A/A(\alpha))) = \frac{\det A}{\det A(\alpha)} \quad (5)$$

In connection with a divide and conquer algorithm for computing the stationary distribution vector for a Markov chain, Meyer [2,3] introduced, for a $n \times n$ nonnegative and irreducible matrix A , the notion of the Perron complement. Again, Let $\alpha \subset \langle n \rangle$ and $\beta = \langle n \rangle \setminus \alpha$. Then the Perron complement of $A(\alpha)$ in A is given by

$$P(A/A(\alpha)) = A(\beta) + A(\beta, \alpha)[\rho(A)I - A(\alpha)]^{-1}A(\alpha, \beta) \quad (6)$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix. Recall that as A is irreducible, $\rho(A) > \rho(A(\alpha))$, so that the expression on the right-hand side of (6) is well defined, and we observe that $\rho(A)I - A(\alpha)$ is an M-matrix and thus $(\rho(A)I - A(\alpha))^{-1} \geq 0$. Meyer [2, 3] has derived several interesting and useful properties of $P(A/A(\alpha))$, such as $P(A/A(\alpha))$ is also nonnegative and irreducible, and $\rho(P(A/A(\alpha))) = \rho(A)$. And such matrices arise in a variety of applications [9], have been studied most of the 20th century, and have received increasing attention of late (see [4,5]). For general irreducible nonnegative matrices, Johnson and Xenophotos [9] investigate when the Perron complements are primitive or just irreducible and thus answer some issues which were raised by Meyer in his earlier paper.

Some of the results in [2, 3, 4, 5] motivated this study on Perron complements of strictly generalized doubly diagonally dominant matrices. In fact, given a matrix family, it is always interesting to know whether some important properties or structures of the family of the matrices are inherited by their submatrices or by the matrices associated with the original matrices.

In addition, for any $\alpha \subset \langle n \rangle$ and for any $t \geq \rho(A)$, let the extended Perron complement at t be the matrix

$$P_t(A/A(\alpha)) = A(\beta) + A(\beta, \alpha)[tI - A(\alpha)]^{-1}A(\alpha, \beta) \quad (7)$$