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Existence of Multiple Solutions for a Class of Nonlinear Elliptic Problems Involving the P-Laplacian

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Abstract. We prove the existence of nontrivial nonnegative solutions to the following nonlinear elliptic problem:

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = \lambda a(x)u^{\alpha-1} + b(x)u^{\beta-1}, x \in \Omega \\ u = 0, x \in \partial\Omega \end{cases}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$, $1 < p < 2$, $\Omega \subset R^n$ is a bounded domain with smooth boundary , $1 < p < 2 < \beta < \alpha < p^*$ ($p^* = \frac{pn}{n-p}$ if $n > p$, $p^* = \infty$ if $n \le p$) , $\lambda \in R \setminus \{0\}$ is a real parameter , the weight $m(x)$ is a bounded function with $\| m \|_\infty > 0$ and $a(x)$, $b(x)$ are continuous functions which change sign in $\overline{\Omega}$.

1. Introduction

We are concerned with the existence and multiplicity of nontrivial nonnegative solutions to the nonlinear elliptic problem:

$$\begin{cases} -\Delta_p u + m(x)u^{p-1} = \lambda a(x)u^{\alpha-1} + b(x)u^{\beta-1}, x \in \Omega \\ u = 0, x \in \partial\Omega \end{cases} \tag{1}$$

where Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$, $1 < p < 2$, $\Omega \subset R^n$ is a bounded domain with smooth boundary, $1 < p < 2 < \beta < \alpha < p^*$, ($p^* = \frac{pn}{n-p}$ if $n > p$, $p^* = \infty$ if $n = p$), $\lambda \in R \setminus \{0\}$, the weight $m(x)$ is a bounded function with $\| m \|_\infty > 0$ and $a(x), b(x) \in C(\overline{\Omega})$ are satisfying $a^\pm = \max\{\pm a, 0\} \not\equiv 0$ and $b^\pm = \max\{\pm b, \infty\} \not\equiv 0$.

Problems involving the "p-Laplacian" arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping (see[8,13]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids: pseudo-plastic fluids correspond to $p \in (1,2)$ while dilatant fluids correspond to $p > 2$. The case $p = 2$ expresses Newtonian fluids [5].

We are motivated by the paper of Wu [14], in which problem (1) was discussed when $m \equiv 1$, $b \equiv 1$, $p = 2$, and $1 < \alpha < 2 < \beta < 2^*$. The authors proved that, there exists $\lambda_0 > 0$ such that if the parameter λ satisfy $0 < \lambda < \lambda_0$, then problem (1) for $m \equiv 1, b \equiv 1, p = 2$ and $1 < \alpha < 2 < \beta < 2^*$, has at least two positive solutions. Using the technique of Brown and Wu [7], in [15] the author discussed problem (1) with $m \not\equiv 1, b \not\equiv 1$, $p > 2$,and $2 < \beta < p < \alpha < p^*$.They obtained at least two positive solutions. In this paper, we discuss the problem (1) with $m \not\equiv 1, b \not\equiv 1$, $1 < p < 2$ and $1 < p < 2 < \beta < \alpha < p^*$. The change in

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α completely changes the nature of the solution set of (1). In fact, we shall prove that the problem (1) has at least two solutions u_0^+ and u_0^- such that $u_0^\pm \geq 0$ in Ω and $u_0^\pm \neq 0$ when the parameter λ belongs to a certain subset of R.

In the case when $p=2$, similar problems (with Dirichlet or Neuman boundary condition) have been studied by Binding et al. [6], Ambrosetti et al. [3], and Tehrani [11,12] , by using variational methods and by Amman and Lopez-Gomez [4] used global bifurcation theory to study the problem. Similar problem in the ODE case (semilinear or quasilinear) have been studied in [1,9]. We refer to [2,10] for additional results on elliptic problems involving the *p*-Laplacian.

2. Variational setting

Let $W_0^{1,s}(\Omega) = W_0^{1,s}, (s>0)$, denote the usual Sobolev space. In the Banach spac $W_0^{1,p}(\Omega) = W$ we introduce the norm

$$\| u \|_W = \left(\int_\Omega \left(|\nabla u|^p + m(x)|u|^p\right)dx\right)^{\frac{1}{p}}$$

which is equivalent to the standard one. First we give the definition of the weak solution of Eq. (1).

Definition 2.1. We say that $u \in W$ is a weak solution to (1) if for any $v \in W$ we have

$$\int_\Omega (|\nabla u|^{p-2}\nabla u.\nabla v + m(x)|u|^{p-2}uv)dx = \lambda\int_\Omega a(x)|u|^{\alpha-2}uvdx + \int_\Omega b(x)|u|^{\beta-2}uvdx$$

It is clear that Problem (1) has a variational structure. Let $J_\lambda : W \to R$ be the corresponding energy functional of problem (1) is defined by

$$J_\lambda(u) = \frac{1}{p}M(u) - \frac{1}{\alpha}A(u) - \frac{1}{\beta}B(u)$$

where

$$M(u) = \int_\Omega (|\nabla u|^p + m(x)|u|^p)dx\ ,\ A(u) = \lambda\int_\Omega a(x)|u|^\alpha dx$$

and

$$B(u) = \int_\Omega b(x)|u|^\beta dx$$

It is well known that the weak solutions of Eq. (1) are the critical points of the energy functional J_λ. Let *I* be the energy functional associated with an elliptic problem on a Banach space *X*. If *I* is bounded below and *I* has a minimizer on *X*, then this minimizer is a critical point of *I*. So, it is a solution of the corresponding elliptic problem. However, the energy functional J_λ, is not bounded below on the whole space *W*, but is bounded on an appropriate subset, and a minimizer on this set (if it exists) gives rise to solution to Eq. (1).

Consider the Nehari minimization problem for $\lambda \in R \setminus \{0\}$,

$$\gamma_\lambda = \inf\{J_\lambda(u) : u \in N_\lambda\},$$

where $N_\lambda = \{u \in W \setminus \{0\} : \langle J'_\lambda(u),(u)\rangle = 0\}$. It is easy to see that $u \in N_\lambda$ if and only if

$$M(u) - A(u) = B(u). \tag{2}$$

Note that N_λ contains every nonzero solution of problem (1). Define

$$g_\lambda(u) = \langle J'_\lambda(u), u\rangle.$$

Then for $u \in N_\lambda$,

$$< g'_\lambda(u), u >= pM(u) - \alpha A(u) - \beta B(u) \tag{3}$$

$$= (p-\alpha)A(u) - (\beta - p)B(u) \tag{4}$$

$$= (p-\alpha)M(u) - (\beta-\alpha)B(u) \tag{5}$$

$$= (p-\beta)M(u) - (\alpha-\beta)A(u). \tag{6}$$