

Computational Method for Nonlinear Singularly Perturbed Singular Boundary Value Problems using Nonpolynomial Spline

Navnit Jha^a

^aDepartment of Mathematics, Rajiv Gandhi Institute of Petroleum Technology Rae Bareilly, U.P, India-229346

(Received May 30, 2011, accepted July 2, 2011)

Abstract. We report a fourth order accurate numerical technique via nonpolynomial spline for singularly perturbed singular two point boundary value problems of the form

$$-\varepsilon u''(r) + f(r, u, u') = 0, \quad u(a) = A, \quad u(b) = B.$$

The numerical scheme is developed for problems arising in the various fields of science and engineering. The scheme is three point nonlinear systems of equations. The method is applied to a few test examples to illustrate the accuracy and the implementation of the method.

Keywords: Non polynomial spline, Singular perturbation, Singular equation, Boundary layer, Taylor's series, Root mean square errors.

1. Introduction

Consider the following nonlinear singular perturbation problems (SPP)

$$-\varepsilon u''(r) + f(r, u, u') = 0, \quad u(a) = A, \quad u(b) = B, \quad a \leq r \leq b \quad (1)$$

where $0 < \varepsilon \ll 1$, A and B are finite constants and assuming that f is bounded and smooth function satisfying

$$\frac{\partial f}{\partial u} \geq 0, \quad \frac{\partial f}{\partial v} \leq 0, \quad \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} \geq \lambda > 0$$

$$\lim_{v \rightarrow \infty} f(r, u, v) = O(|v|^2), \quad a \leq r \leq b \quad \text{and} \quad u, v \in R$$

Howes [1], suggested that under the above conditions, the problem (1) posses a unique solution. SPP occur in many branches of science and engineering such as heat transport problems with large Peclet numbers, Navier-Stokes flows with large Reynolds numbers, convection-diffusion process, gas porous electrodes theory, fluid dynamics, chemical kinetics, modeling of steady and unsteady viscous flow problems. The solution of SPP exhibits a multi-scale character. There are many methods based on finite difference, boundary element, collocations method etc. available for solving linear SPP[2-10]. Recently, Tirmizi [11], have proposed a nonpolynomial spline method for linear singular perturbation problems which has second and fourth order of convergence depending upon the choice of free parameters. Kadalbajoo and Patidar [12] has considered second order convergent spline in compression technique for the nonlinear singular perturbation problems. However, their methods are only applicable to non-singular problems. Difficulties were experienced in the past for the numerical solution of singularly perturbed singular two point boundary value problems in polar coordinates. The solution usually deteriorates in the vicinity of singularity. The aim of this paper is to design a computationally efficient numerical technique based on nonpolynomial spline and finite difference approximations in such a way that fourth order convergence is retained for smaller values of ε and restriction on grid size can be avoided in case of singularity.

In this paper, we are concerned with the problem of applying nonpolynomial spline functions to develop numerical schemes for obtaining approximate solution for the nonlinear singular two point boundary value

^a Email address: navnitjha@rgipt.ac.in

problems. The C^∞ - differentiability of the trigonometric part of nonpolynomial spline basis compensates for the loss of smoothness inherited by polynomial splines. The resulting nonpolynomial spline three point difference schemes are of fourth order accuracy. The importance of our work is that the proposed methods are applicable to problems both in rectangular and polar coordinates.

The paper is organized as follows: In section 2, we give a brief description of the mathematical method. In section 3, we design difference schemes of class of singular equation in operator compact form. Some nonlinear singular and nonsingular examples are illustrated to justify the accuracy and efficiency of the proposed method in section 4. The numerical results exhibit oscillation free solution for $0 < \varepsilon \ll 1$, even in the vicinity of the singularity.

2. Nonpolynomial Spline Finite Difference Method

For the numerical approximation of problems (1), we divided the domain $\Omega = [a, b]$ into a set of nodes with interval of $h = 1/(N+1)$, N being a positive integer. The nonpolynomial spline approximations is obtained on Ω that consists of the central point $r_k = a + kh$ and two neighbouring grids $r_{k\pm 1}$. The approximate solution of this equation is sought in the form of the function $S_k(r)$ which interpolates $f(r, u, u')$ at r_k defined as follows

$$S_k(r) = \alpha_k \sin \tau(r - r_k) + \beta_k \cos \tau(r - r_k) + \gamma_k(r - r_k) + \delta_k, \quad k = 0(1)N \quad (2)$$

where α_k , β_k , γ_k , and δ_k are constants and τ is the frequency of the trigonometric functions. Thus, the cubic nonpolynomial spline is defined by the relations:

$$(i) \quad S(r) \in C^\infty(\Omega) \quad (3)$$

$$(ii) \quad S''(r_k) = M_k, \quad S(r_k) = u_k, \quad k = 0(1)N+1$$

We obtain via algebraic calculations the following expressions

$$\alpha_k = \frac{h^2}{\theta^2 \sin \theta} (M_k \cos \theta - M_{k+1}), \quad \beta_k = -\frac{h^2}{\theta^2} M_k$$

$$\gamma_k = \frac{1}{h} (u_{k+1} - u_k) + \frac{h}{\theta^2} (M_{k+1} - M_k), \quad \delta_k = u_k + \frac{h^2}{\theta^2} M_k$$

where $\theta = h\tau$.

Following, Islam and Tirmizi [11, 13] and, Rashidinina *et. al.* [14], we obtain

$$u_{k-1} - 2u_k + u_{k+1} - h^2(\alpha M_{k-1} + 2\beta M_k + \alpha M_{k+1}) = 0, \quad k = 1(1)N \quad (4)$$

where, $\alpha = \frac{\theta - \sin \theta}{\theta^2 \sin \theta}$, $\beta = \frac{\sin \theta - \theta \cos \theta}{\theta^2 \sin \theta}$

Consider the following approximations

$$\hat{u}'_k = \frac{u_{k+1} - u_{k-1}}{2h}, \quad \hat{u}'_{k\pm 1} = \frac{\pm 3u_{k\pm 1} \mp 4u_k \pm u_{k\mp 1}}{2h}, \quad \hat{H}_{k\pm 1} = f(r_{k\pm 1}, u_{k\pm 1}, \hat{u}'_{k\pm 1})$$

$$\hat{\hat{u}}'_k = \hat{u}'_k + h\omega(\hat{H}_{k+1} - \hat{H}_{k-1}) \quad (5)$$

The above nonpolynomial spline finite difference approximation for $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, have local

truncation errors of $-\frac{h^4}{12}(1 + 20\varepsilon\omega)u'''(r_k) + O(h^6)$.

3. Application to Singular Problems

We discuss the application of method (4), for the numerical solution of model problems of various classes of singular perturbation problems. Consider the following singularly perturbed model problem