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Some Comparative Growth Rate of Composite Entire and Meromorphic Functions

Sanjib Kumar Datta [2] and Tanmay Biswas [1]

[1] Department of Mathematics, University of Kalyani, Kalyani, Dist-Nadia, PIN- 741235, West Bengal, India.

[2] Rajbari, Rabindrapalli, R. N. Tagore Road, P.O. Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India.

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Abstract. In this paper we discuss some growth rates of composite entire and meromorphic functions improving some earlier results.

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1. Introduction, Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be a meromophic function and g be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [6] and [3].

In the sequel we use the following notation:

$$log^{[k]}x = log(log^{[k-1]}x) \text{ for } k = 1,2,3,\dots \text{ and}$$
$$log^{[0]}x = x\,;$$

and

$$exp^{[k]}x = exp(exp^{[k-1]}x) \text{ for } k = 1,2,3,\dots \text{ and}$$
$$exp^{[0]}x = x\,.$$

The following definitions are well known.

Definition 1 The order ρ_f and lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r} \qquad and \qquad \lambda_f = \liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}.$$

If f is meromorphic then

$$\rho_f = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log r} \qquad and \qquad \lambda_f = \liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

Juneja, Kapoor and Bajpai [4] defined the (p,q) th order and (p,q) th lower order of an entire function f respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}r} \qquad and \qquad \lambda_f(p,q) = \liminf_{r\to\infty}\frac{\log^{[p]}M(r,f)}{\log^{[q]}r},$$

where p, q are positive integers and $p > q$.

When f is meromorphic, one can easily verify that

[1] *E-mail address*: sanjib_kr_datta@yahoo.co.in;
[2]*E-mail address*: Tanmaybiswas_math@rediffmail.com; Tanmaybiswas_math@yahoo.com

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \qquad and \qquad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

where p, q are positive integers and $p > q$.

If $p = 2$ and $q = 1$ then we write $\rho_f(1,2) = \rho_f$ and $\lambda_f(1,2) = \lambda_f$.

The following definitions are also well known.

Definition 2 A meromorphic function $a \equiv a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$.

Definition 3 Let $a_1, a_2, \ldots, a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, \ldots, a_k, f)$ the Wronskian determinant of $a_1, a_2, \ldots, a_k, f$ i.e,

$$L(f) = \begin{vmatrix} a_1 & a_2 & \cdots & a_k & f \\ a_1' & a_2' & \cdots & a_k' & f' \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_1^k & a_2^k & \cdots & a_k^k & f^k \end{vmatrix}.$$

Definition 4 If $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}$$

is called the Nevanlinna's deficiency of the value 'a'.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum_{a \neq \infty} \delta(a;f) + \delta(\infty;f) \le 2$ {cf. [3], p.43}. If in particular, $\sum_{a \neq \infty} \delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximum deficiency sum.

In the paper we establish some newly developed results based on the comparative growth properties of composite entire or meromorphic functions and wronskians generated by one of the factors on the basis of $(p; q)$ th order and $(p; q)$ th lower order where p, q are positive integers with $p > q$. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [6] and [3].

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [1] Let f be meromorphic and g be entire then for all sufficiently large values of r,

$$T(r, f \circ g) \le \{1 + \circ(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma 2 [2] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \le \infty$. Then for a sequence of values of r tending to infinity,

$$T(r, f \circ g) \ge T(\exp(r^\mu), f).$$

Lemma 3 [5] Let f be a transcendental meromorphic function having the maximum deficiency sum. Then

$$\lim_{r\to\infty} \frac{T(r, L(f))}{T(r,f)} = 1 + k - k\delta(\infty; f).$$

Lemma 4 If f be a transcendental meromorphic function with the maximum deficiency sum, then the (p,q) th order and (p,q) th lower order of $L(f)$ are same as those of f.

Proof. By Lemma 3,

$$\lim_{r\to\infty} \frac{\log^{[p-1]} T(r, L(f))}{\log^{[p-1]} T(r,f)}$$

where p is any positive integer > 1 exists and is equal to 1. So

$$\rho_{L(f)}(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r, L(f))}{\log^{[q]} r}$$