

A collocation method for the solution of convection–diffusion parabolic problems

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Abstract. In this paper, we develop a collocation method based on cubic B-spline to the solution of singularly perturbed parabolic equation $\varepsilon^2 \frac{\partial^2 u}{\partial x^2} - c(x,t)u - p(x,t) \frac{\partial u}{\partial t} = f(x,t)$, subject to appropriate initial, and Dirichlet boundary conditions, where ε^2 , is a small constant. We develop a new two-level three-point scheme of order $O(k^2 + h^2)$. The convergence analysis of the method is proved. Numerical results are given to illustrate the efficiency of our method computationally.

Keywords: B-Spline, parabolic equation, convergence, singularly perturbed..

1. Introduction

In this paper, we consider singularly perturbed parabolic equation :

$$\varepsilon^2 \frac{\partial^2 u}{\partial x^2} - c(x,t)u - p(x,t) \frac{\partial u}{\partial t} = f(x,t), \quad (1)$$

on the domain D , where $D = (0,1) \times (0,T]$,
with initial condition

$$u(x,0) = u_0(x), \quad x \in (0,1), \quad (2)$$

and boundary conditions

$$u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad t \in (0,T], \quad (3)$$

We assume that

1. functions $c(x,t)$, $p(x,t)$ and $f(x,t)$ are sufficiently smooth on the D , and $c(x,t) \geq 0$, $p(x,t) \geq p_0 > 0$, $(x,t) \in D$
2. functions $g_0(t)$ and $g_1(t)$ are sufficiently smooth on the $[0,T]$ and $u_0(x)$ is smooth on $[0,1]$,
3. $\varepsilon \in (0,1]$, and
4. Compatibility conditions are satisfied at the corner points $(0,0)$ and $(1,0)$.

We suppose that $\varepsilon' \equiv \varepsilon^2$, $\kappa(x,t,u,u_x) = c(x,t)u + f(x,t) \Rightarrow \varepsilon' u_{xx} = p(x,t)u_t + \kappa(x,t,u,u_x)$

In this paper we have developed two-level implicit difference scheme by using cubic B-spline function for the solution of singularly perturbed parabolic problem (1). This paper is arranged as follows. In section2, we present a finite difference approximation to discretize the equation (1) and obtain the convergence of method in time variable. In section3 we applied cubic B-spline collocation method to solve the ordinary differential equations obtained in each time level. The uniform convergence of the method is proved in section4. In section5, numerical experiments are conducted to demonstrate the viability and the efficiency of the proposed method computationally.

2. Temporal discretization

Let us consider a uniform mesh Δ with the grid points $\lambda_{i,j}$ to discretize the region $\Omega = (0,1) \times (0,T]$. Each $\lambda_{i,j}$ is the vertices of the grid point (x_i, t_j) where $x_i = ih, i = 0,1,2,\dots,N$ and $t_j = jk, j = 0,1,2,\dots$ and h and k are mesh sizes in the space and time directions respectively.

First we use the following finite difference approximation to discretize the time variable with uniform step size k ,

$$u_t^n \cong \frac{\delta_t}{k(1-\gamma\delta_t)} u^n, \quad n \geq 0, \quad \gamma \neq 1, \quad (4)$$

where n is the step number in t direction and $\delta_t u^n = u^n - u^{n-1}$, $u^n = u(x, t_n)$ and $u^0 = u(x, 0) = u_0(x)$, $(0 < x < 1)$.

Substituting the above approximation into equation (1) and discretizing in time variable we have:

$$\varepsilon' u_{xx}^n = p(x, t_n) \frac{\delta_t}{k(1-\gamma\delta_t)} u^n + \kappa(x, t_n, u^n, u_x^n), \quad (5)$$

thus we have,

$$\varepsilon' k(1-\gamma\delta_t) u_{xx}^n = p(x, t_n) \delta_t u^n + k(1-\gamma\delta_t) \kappa(x, t_n, u^n, u_x^n), \quad (6)$$

now by simplifying we can write equation (6) in the following form

$$-\varepsilon' u_{xx}^* + p^*(x) u^* + \kappa^*(x, u^*, u_x^*) = q^*(x), \quad (7)$$

where $u^* \equiv u^n$ and,

$$\begin{cases} u_{xx}^* \equiv u_{xx}^n, \\ p^*(x) = \frac{p(x, t_n)}{k(1-\gamma)}, \\ q^*(x) = \frac{p(x, t_n)}{k(1-\gamma)} u^{n-1} + \frac{\varepsilon' \gamma}{1-\gamma} u_{xx}^{n-1} - \frac{\gamma}{1-\gamma} \kappa(x, t_n, u^{n-1}, u_x^{n-1}), \\ \kappa^*(x, u^*, u_x^*) = \kappa(x, t_n, u^n, u_x^n). \end{cases}$$

with boundary conditions,

$$u^*(0) = g_0(t_n), \quad u^*(1) = g_1(t_n). \quad (8)$$

Thus now in each time level we have a nonlinear ordinary differential equation in the form of (7) with the boundary conditions (8) which can be solved by using B-spline collocation method.

Theorem 2.1:

The above time discretization process that we use to discretize equation (1) in time variable is of the second order convergence.

Proof:

Let $u(t_i)$ be the exact solution and u^i the approximate solution of the problem (1) at the i th level time and also suppose that $e_i = u^i - u(t_i)$ be the local truncation error in (7). Then using equation (4) and replacing $\gamma = \frac{1}{2}$ it can be easily proved that,

$$|e_i| \leq c_i k^3, \quad (9)$$

where c_i is some finite constant.

Let E_{n+1} be the global error in time discretizing process then the global error in $(n+1)$ th level is

$$E_{n+1} = \sum_{i=1}^n e_i, \quad \left(t \leq \frac{T}{n} \right)$$

thus with the help of (9) we have: