

Numerical solution of the boundary value problems in calculus of variations using parametric cubic spline method

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Abstract. In this paper, a numerical solution based on parametric cubic spline is used for finding the solution of boundary value problems arising in the calculus of variations. The present approach has less computational coast and gives better approximation. This approximation reduce the problems to an explicit system of algebraic equations. Some numerical examples are also given to illustrate the accuracy and applicability of the presented method.

Keywords: Calculus of variation, Parametric spline, Numerical method.

1. Introduction

The calculus of variations and its extensions are devoted to finding the optimum function that gives the best value of the economic model and satisfies the constraints of a system. The need for an optimum function, rather than an optimal point, arises in numerous problems from a wide range of fields in engineering and physics, which include optimal control, transport phenomena, optics, elasticity, vibrations, statics and dynamics of solid bodies and navigation[1]. In computer vision the calculus of variations has been applied to such problems as estimating optical flow [2] and shape from shading [3]. Several numerical methods for approximating the solution of problems in the calculus of variations are known. Galerkin method is used for solving variational problems in [4]. The Ritz method [5], usually based on the subspaces of kinematically admissible complete functions, is the most commonly used approach in direct methods of solving variational problems. Chen and Hsiao [6] introduced the Walsh series method to variational problems. Due to the nature of the Walsh functions, the solution obtained was piecewise constant. Some orthogonal polynomials are applied on variational problems to find the continuous solutions for these problems [7-9]. A simple algorithm for solving variational problems via Bernstein orthonormal polynomials of degree six is proposed by Dixit et al. [10]. Razzaghi et al. [11] applied a direct method for solving variational problems using Legendre wavelets. Adomian decomposition method has been employed for solving some problems in calculus of variations in [12].

Spline functions are special functions in the space of which approximate solutions of ordinary differential equations. In other words spline function is a piecewise polynomial satisfying certain conditions of continuity of the function and its derivatives. The applications of spline as approximating, interpolating and curve fitting functions have been very successful[13-16]. In [17], a cubic non-polynomial spline technique has been developed for the numerical solutions of a system of fourth order boundary value problems associated with obstacle, unilateral and contact problems. Quadratic and cubic polynomial and non-polynomial spline functions based methods have been presented to find approximate solutions to second order boundary value problems[18]. Parametric spline method for a class of singular two-point boundary value problems has been developed by Rashidinia et al. [19]. The main purpose of the present paper is to use parametric cubic spline method for numerical solution of boundary value problems

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which arise from problems of calculus of variations. The method consists of reducing the problem to a set of algebraic equations.

The outline of the paper is as follows. First, in Section 2 we introduce the problems in calculus of variations and explain their relations with boundary value problems. Section 3 outlines parametric cubic spline and basic equations that are necessary for the formulation of the discrete system. Also in this section, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering two numerical examples.

2. Statement of the problem

The general form of a variational problem is finding extremum of the functional

$$J[u_1(t), u_2(t), \dots, u_n(t)] = \int_a^b G(t, u_1(t), u_2(t), \dots, u_n(t), u'_1(t), u'_2(t), \dots, u'_n(t)) dt. \quad (1)$$

To find the extreme value of J , the boundary conditions of the admissible curves are known in the following form:

$$u_i(a) = \gamma_i, \quad i = 1, 2, \dots, n, \quad (2)$$

$$u_i(b) = \delta_i, \quad i = 1, 2, \dots, n. \quad (3)$$

The necessary condition for $u_i(t)$, $i = 1, 2, \dots, n$ to extremize $J[u_1(t), u_2(t), \dots, u_n(t)]$ is to satisfy the Euler-Lagrange equations that is obtained by applying the well known procedure in the calculus of variation [5],

$$\frac{\partial G}{\partial u_i} - \frac{d}{dt} \left(\frac{\partial G}{\partial u'_i} \right) = 0, \quad i = 1, 2, \dots, n, \quad (4)$$

subject to the boundary conditions given by Eqs. (2)-(3).

In this paper, we consider the special forms of the variational problem (1) as

$$J[u(t)] = \int_a^b G(t, u(t), u'(t)) dt, \quad (5)$$

with boundary conditions

$$u(a) = \gamma, \quad u(b) = \delta, \quad (6)$$

and

$$J[u_1(t), u_2(t)] = \int_a^b G(t, u_1(t), u_2(t), u'_1(t), u'_2(t)) dt, \quad (7)$$

subject to boundary conditions

$$u_1(a) = \gamma_1, \quad u_1(b) = \delta_1, \quad (8)$$

$$u_2(a) = \gamma_2, \quad u_2(b) = \delta_2. \quad (9)$$

Thus, for solving the variational problems (5), we consider the second-order differential equation

$$\frac{\partial G}{\partial u} - \frac{d}{dt} \left(\frac{\partial G}{\partial u'} \right) = 0, \quad (10)$$

with the boundary condition (6). And also, for solving the variational problems (7), we find the solution of the system of second-order differential equations