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Relative Order of Functions of Several Complex Variables Analytic in the Unit Polydisc

Ratan Kumar Dutta

Department of Mathematics, Patulia High School, Patulia, Kolkata-700119, West Bengal, India
E-mail: ratan_3128@yahoo.com

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Abstract. Throughout the paper we consider relative order of functions of several complex variables analytic in the unit poly disc with respect to an entire function and after proving several theorems, we show that relative order of analytic function and its partial derivatives are same.

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1. Introduction

A function f analytic in the unit disc $U:\{z:|z|<1\}$, is said to be of finite Nevanlinna order [7] (Juneja and Kapoor 1985) if there exists a number μ such that Nevanlinna characteristic function $T(r,f)$ of f defined by

$$T(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta$$

satisfies

$$T(r,f)=(1-r)^{-\mu}$$

for all r in $0<r_0(\mu)<r<1$.

The greatest lower bound of all such numbers μ is called Nevanlinna order of f. Thus the Nevanlinna order $\rho(f)$ of f is given by

$$\rho(f)=\limsup_{r\to 1}\frac{\log T(r,f)}{-\log(1-r)}.$$

In [1] Banerjee and Dutta introduced the idea of relative order of an entire function which as follows:

Definition 1.1. If f be analytic in U and g be entire, then the relative order of f with respect to g, denoted by $\rho_g(f)$ is defined by

$$\rho_g(f)=\inf\{\mu>0:T_f(r)<T_g\left[\left(\frac{1}{1-r}\right)^{\mu}\right]\text{for all } 0<r_0(\mu)<r<1\}.$$

Note 1.2. When $g(z)=\exp z$ then the Definition 1.1 coincides with the definition of Nevanlinna order of f.

Also in [2] Banerjee and Dutta introduced the idea of relative order of an entire function of two complex variables which as follows:

Definition 1.3. Let $f(z_1,z_2)$ be a non-constant analytic function of two complex variables z_1 and z_2 holomorphic in the closed unit poly disc $P:\{(z_1,z_2):|z_j|\le 1;\ j=1,2\}$ and $g(z_1,z_2)$ be an entire function then relative order of f with respect to g denoted by $\rho_g(f)$ and is defined by

$$\rho_g(f)=\inf\{\mu>0:F(r_1,r_2)<G\left(\frac{1}{(1-r_1)^\mu},\frac{1}{(1-r_2)^\mu}\right)\text{for all } 0<r_0(\mu)<r_1,r_2<1\}.$$

In a resent paper [3] Dutta introduced the following definition.

Definition 1.4. Let $f(z_1,z_2.....z_n)$ and $g(z_1,z_2.....z_n)$ be two entire functions of n complex variables $z_1,z_n.....z_n$ with maximum modulus functions $F(r_1,r_2.....r_n)$ and $G(r_1,r_2.....r_n)$ respectively then relative order of f with respective to g, denoted by $\rho_g(f)$ and is defined by

$$\rho_g(f)=\inf\{\mu>0:F(r_1,r_2.....r_n)<G(r_1^\mu,r_2^\mu.....r_n^\mu)\ for\ r_i\geq R(\mu);i=1,2......n\}.$$

Also in a paper [4] Dutta introduced the following definition.

Definition 1.5. Let $f(z_1,z_2,......z_n)=\sum_{m_1,m_2,......m_n=0}^{\infty}c_{m_1m_2......m_n}z_1^{m_1}z_2^{m_2}......z_n^{m_n}$ be a function of n complex variables $z_1,z_2,......z_n$ holomorphic in the unit polydisc

$$P=\{(z_1,z_2,......z_n):|z_j|\leq 1;j=1,2,......n\}$$

and

$$F(r_1,r_2,......r_n)=\max\{|f(z_1,z_2,......z_n)|:|z_j|\leq r_j;j=1,2,......n\},$$

be its maximum modulus. Then the order ρ and lower order λ are defined as

$$\begin{matrix}\rho\\ \lambda\end{matrix}=\lim_{r_1,r_2,......r_n\to 1}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log F(r_1,r_2,......r_n)}{-\log(1-r_1)(1-r_2)......(1-r_n)}.$$

Now we introduce the following definition.

Definition 1.6. Let $f(z_1,z_2.....z_n)$ be a non-constant analytic function of several complex variables $z_1,z_n.....z_n$ holomorphic in the closed unit polydisc

$$P:\{(z_1,z_2.....z_n):|z_j|\leq 1;j=1,2.....n\}$$

and $g(z_1,z_2.....z_n)$ be an entire function then relative order of f with respect to g denoted by $\rho_g(f)$ and defined by

$$\rho_g(f)=\inf\{\mu>0:F(r_1,r_2.....r_n)<G\left(\frac{1}{(1-r_1)^\mu},\frac{1}{(1-r_2)^\mu}......\frac{1}{(1-r_n)^\mu}\right)$$
$$\text{for all } 0<r_0(\mu)<r_1,r_2.....r_n<1\}$$

where $G(r_1,r_2.....r_n)=\max\{|g(z_1,z_2.....z_n)|:|z_j|=r_j;j=1,2.....n\}$.

Note 1.7. When $g(z_1,z_2.....z_n)=e^{z_1z_2.....z_n}$ then Definition 1.6 coincides with the Definition 1.5 and if n=2 then coincide with Definition 1.3.

We require the following definition.

Definition 1.8. An entire function $g(z_1,z_2.....z_n)$ is said to have the property (R) if for any $\sigma>1,\lambda>0$ and for all r_i sufficiently close to 1; $i=1,2.....n,$

$$\left[G\left(\frac{1}{(1-r_1)^\lambda},\frac{1}{(1-r_2)^\lambda}......\frac{1}{(1-r_n)^\lambda}\right)\right]^2<G\left(\frac{1}{((1-r_1)^\lambda)^\sigma},\frac{1}{((1-r_2)^\lambda)^\sigma}......\frac{1}{((1-r_n)^\lambda)^\sigma}\right).$$

Note 1.9. The function $g(z_1,z_2.....z_n)=e^{z_1z_2.....z_n}$ has the property (R) but $g(z_1,z_2.....z_n)=z_1z_2.....z_n$ has not.