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Growth of Iterated Entire Functions in terms of (p, q)-th Order

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Abstract. In this paper we discuss some growth rates of iterated entire functions improving some earlier results.

Keywords: Entire functions, growth, iteration, order, lower order, (p,q)-th order, lower (p,q)-th order.

1. Introduction, Definitions and Notation

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in the open complex plane **C**. It is well known [1], {[15], p-67, Th-1.46} that

$$\lim_{r\to\infty}\frac{T(r, fog)}{T(r, f)}=\infty \quad \text{and} \quad \lim_{r\to\infty}\frac{T(r, fog)}{T(r, g)}=\infty .$$

After this Singh [11], Lahiri [7], Song and Yang [13], Singh and Baloria [12], Lahiri and Sharma [8] and Datta and Biswas [3], [4] proved different results on comparative growth property of composite entire functions. In a resent paper [2] Dutta study some comparative growth of iterated entire functions. In this paper, we investigate the comparative growth of iterated entire functions in terms of its (p,q)-th order. We do not explain the standard notations and definitions of the theory of entire functions as those are available in [5], [14] and [15].

The following definitions are well known.

Definition 1.1. The order ρ_f and lower order λ_f of a meromorphic function $f(z)$ is defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log T(r, f)}{\log r}$$

and

$$\lambda_f=\liminf_{r\to\infty}\frac{\log T(r, f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f=\limsup_{r\to\infty}\frac{\log\log M(r, f)}{\log r}$$

and

$$\lambda_f=\liminf_{r\to\infty}\frac{\log\log M(r, f)}{\log r}.$$

Notation 1.2. [10] $\log^{[0]}x=x, \exp^{[0]}=x$ and for positive integer m, $\log^{[m]}x=\log(\log^{[m-1]}x),\ \exp^{[m]}x=\exp(\exp^{[m-1]}x)$.

Definition 1.3. The p-th order ρ_f^p and lower p-th order λ_f^p of a meromorphic function $f(z)$ is defined as

$$\rho_f^p=\limsup_{r\to\infty}\frac{\log^{[p]}T(r, f)}{\log r}$$

and

$$\lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f^p = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}$$

and

$$\lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}.$$

Clearly $\rho_f^p \le \rho_f^{p-1}$ and $\lambda_f^p \le \lambda_f^{p-1}$ for all *p* and when *p=1* then p-th order and lower p-th order coincide with classical order and lower order respectively.

Definition 1.4. The (p,q) -th order $\rho_f(p,q)$ and lower (p,q)-th order $\lambda_f(p,q)$ of a meromorphic function $f(z)$ is define as

$$\rho_f(\mathrm{p,q}) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}$$

and

$$\lambda_f(\mathrm{p,q}) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[q]} r}.$$

If $f(z)$ is an entire function then

$$\rho_f(\mathrm{p,q}) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}$$

and

$$\lambda_f(\mathrm{p,q}) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log^{[q]} r}$$

where $p \ge q \ge 1$.

Clearly $\rho_f(p,1) = \rho_f^p \quad and \quad \lambda(p,1) = \lambda_f^p.$

Definition 1.5. Let $f(z)$ be an entire function of finite p- th order ρ_f^p then we define σ_f^p as

$$\sigma_f^p = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{r^{\rho_f^p}}.$$

According to Lahiri and Banerjee [6] if $f(z)$ *and* $g(z)$ are entire functions then the iteration of f with respect to g is defined as follows:

$$f_1(z) = f(z)$$

$$f_2(z) = f(g(z)) = f\left(g_1(z)\right)$$

$$f_3(z) = f\left(g(f(z))\right) = f\left(g_2(z)\right) = f\left(g\left(f_1(z)\right)\right)$$

$$\cdots\cdots\quad \cdots\cdots\quad \cdots\cdots\quad \cdots\cdots$$

$$f_n(z) = f(g(f\ldots\ldots\ldots\ldots(f(z)\ or\ \ g(z))\ldots\ldots\ldots)),$$

according as *n* is odd or even,