

Growth of Iterated Entire Functions in terms of (p, q)-th Order

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Abstract. In this paper we discuss some growth rates of iterated entire functions improving some earlier results.

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1. Introduction, Definitions and Notation

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in the open complex plane $\mathbb{C}$. It is well known [1], [15], p-67, Th-1.46} that

$$\lim_{r \rightarrow \infty} \frac{T(r, fog)}{T(r, f)} = \infty \quad \text{and} \quad \lim_{r \rightarrow \infty} \frac{T(r, fog)}{T(r, g)} = \infty.$$

After this Singh [11], Lahiri [7], Song and Yang [13], Singh and Baloria [12], Lahiri and Sharma [8] and Datta and Biswas [3], [4] proved different results on comparative growth property of composite entire functions. In a recent paper [2] Dutta study some comparative growth of iterated entire functions. In this paper, we investigate the comparative growth of iterated entire functions in terms of its (p,q)-th order. We do not explain the standard notations and definitions of the theory of entire functions as those are available in [5], [14] and [15].

The following definitions are well known.

Definition 1.1. The order ρ_f and lower order λ_f of a meromorphic function $f(z)$ is defined as

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}.$$

Notation 1.2. $\log^{[0]} x = x$, $\exp^{[0]} x = x$ and for positive integer

$$m, \log^{[m]} x = \log(\log^{[m-1]} x), \quad \exp^{[m]} x = \exp(\exp^{[m-1]} x).$$

Definition 1.3. The p-th order ρ_f^p and lower p-th order λ_f^p of a meromorphic function $f(z)$ is defined as

$$\rho_f^p = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} T(r, f)}{\log r}$$

and

$$\lambda_f^p = \liminf_{r \rightarrow \infty} \frac{\log^{[p]} T(r, f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f^p = \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f)}{\log r}$$

and

$$\lambda_f^p = \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f)}{\log r}.$$

Clearly $\rho_f^p \leq \rho_f^{p-1}$ and $\lambda_f^p \leq \lambda_f^{p-1}$ for all p and when $p=1$ then p -th order and lower p -th order coincide with classical order and lower order respectively.

Definition 1.4. The (p, q) -th order $\rho_f(p, q)$ and lower (p, q) -th order $\lambda_f(p, q)$ of a meromorphic function $f(z)$ is define as

$$\rho_f(p, q) = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} r}$$

and

$$\lambda_f(p, q) = \liminf_{r \rightarrow \infty} \frac{\log^{[p]} T(r, f)}{\log^{[q]} r}.$$

If $f(z)$ is an entire function then

$$\rho_f(p, q) = \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} r}$$

and

$$\lambda_f(p, q) = \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f)}{\log^{[q]} r}$$

where $p \geq q \geq 1$.

Clearly $\rho_f(p, 1) = \rho_f^p$ and $\lambda_f(p, 1) = \lambda_f^p$.

Definition 1.5. Let $f(z)$ be an entire function of finite p -th order ρ_f^p then we define σ_f^p as

$$\sigma_f^p = \limsup_{r \rightarrow \infty} \frac{\log^{[p]} M(r, f)}{r \rho_f^p}.$$

According to Lahiri and Banerjee [6] if $f(z)$ and $g(z)$ are entire functions then the iteration of f with respect to g is defined as follows:

$$f_1(z) = f(z)$$

$$f_2(z) = f(g(z)) = f(g_1(z))$$

$$f_3(z) = f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))$$

$$\dots\dots\dots$$

$$f_n(z) = f(g(f(\dots\dots\dots(f(z) \text{ or } g(z))\dots\dots\dots))),$$

according as n is odd or even,

and so

$$\begin{aligned} g_1(z) &= g(z) \\ g_2(z) &= g(f(z)) = g(f_1(z)) \\ g_3(z) &= g(f_2(z)) = g(f(g(z))) \\ &\dots\dots\dots \\ g_n(z) &= g(f_{n-1}(z)) = g(f(g_{n-2}(z))). \end{aligned}$$

Clearly all $f_n(z)$ and $g_n(z)$ are entire functions.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [5] Let $f(z)$ be an entire function. For $0 \leq r < R < \infty$, we have

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R+r}{R-r} T(R, f).$$

Lemma 2.2. [1] If $f(z)$ and $g(z)$ are any two entire functions, for all sufficiently large values of r ,

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, fog) \leq M(M(r, g), f).$$

Lemma 2.3. [9] Let $f(z)$ and $g(z)$ be two entire functions. Then we have

$$T(r, fog) \geq \frac{1}{3} \log M\left(\frac{1}{8}M\left(\frac{r}{4}, g\right) + O(1), f\right).$$

Lemma 2.4. Let $f(z)$ and $g(z)$ be two entire functions of non zero finite (p, q) -th order $\rho_f(p, q)$

and $\rho_g(p, q)$ respectively, then for any $\varepsilon > 0$ and $p \geq q \geq 1$,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \leq \begin{cases} (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) & \text{when } n \text{ is even,} \\ (\rho_g(p, q) + \varepsilon) \log^{[q]} M(r, f) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of r .

Proof. First suppose that n is even. Then from second part of Lemma 2.2 and Definition of (p, q) -th order, it follows that for all sufficiently large values of r ,

$$M(r, f_n) \leq M\left(M(r, g_{n-1}), f\right)$$

$$\begin{aligned} \text{i.e. , } \log^{[p+1]} M(r, f_n) &\leq \log^{[p+1]} M(M(r, g_{n-1}), f) \\ &\leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g_{n-1}). \end{aligned}$$

$$\text{So, } \log^{[p+2]} M(r, f_n) \leq \log^{[q+1]} M(r, g(f_{n-2})) + o(1)$$

$$\log^{[p+2-q]} M(r, f_n) \leq \log M(r, g(f_{n-2})) + O(1).$$

Taking repeated logarithms p times, we get

$$\begin{aligned}\log^{[2p+2-q]} M(r, f_n) &\leq \log^{[p+1]} M(M(r, f_{n-2}), g) + O(1) \\ &\leq (\rho_g(p, q) + \varepsilon) \log^{[q]} M(r, f_{n-2}) + O(1), \\ \text{i.e., } \log^{[2p+3-q]} M(r, f_n) &\leq \log^{[q+1]} M(r, f_{n-2}) + O(1) \\ \log^{[2p+3-2q]} M(r, f_n) &\leq \log M(r, f_{n-2}) + O(1).\end{aligned}$$

Again taking repeated logarithms p times, we get

$$\log^{[3(p+1)-2q]} M(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g_{n-3}) + O(1).$$

Finally, after taking repeated logarithms $(n-4)(p+1)$ times more, we have for all sufficiently large values of r ,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1).$$

Similarly if n is odd then for all sufficiently large values of r ,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \leq (\rho_g(p, q) + \varepsilon) \log^{[q]} M(r, f) + O(1).$$

This proves the lemma.

Lemma 2.5. Let $f(z)$ and $g(z)$ be two entire functions of non zero finite lower (p, q) -th order

$\lambda_f(p, q)$ and $\lambda_g(p, q)$ respectively, then for any

$0 < \varepsilon < \min\{\lambda_f(p, q), \lambda_g(p, q)\}$ and $p \geq q \geq 1$,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \geq \begin{cases} (\lambda_f(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) + O(1) & \text{when } n \text{ is even} \\ (\lambda_g(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{2^{n-1}}, f\right) + O(1) & \text{when } n \text{ is odd,} \end{cases}$$

for all sufficiently large values of r .

Proof. First suppose that n is even. Then from first part of Lemma 2.2 we have for all sufficiently large values of r and for any $0 < \varepsilon < \min\{\lambda_f(p, q), \lambda_g(p, q)\}$,

$$\begin{aligned}M(r, f_n) &= M\left(r, f(g_{n-1})\right) \\ &\geq M\left(\frac{1}{8} M\left(\frac{r}{2}, g_{n-1}\right) - |g_{n-1}(0)|, f\right) \\ &\geq M\left(\frac{1}{16} M\left(\frac{r}{2}, g_{n-1}\right), f\right).\end{aligned}$$

$$\therefore \log^{[p+1]} M(r, f_n) \geq \left(\lambda_f(p, q) - \varepsilon\right) \log^{[q]} \left[\frac{1}{16} M\left(\frac{r}{2}, g_{n-1}\right)\right],$$

using the Definition 1.4,

$$\text{i.e., } \log^{[p+1]} M(r, f_n) \geq \left(\lambda_f(p, q) - \varepsilon \right) \log^{[q]} M\left(\frac{r}{2}, g_{n-1}\right) + O(1)$$

$$\text{i.e., } \log^{[p+2]} M(r, f_n) \geq \log^{[q+1]} M\left(\frac{r}{2}, g(f_{n-2})\right) + O(1)$$

$$\text{i.e., } \log^{[p+2-q]} M(r, f_n) \geq \log M\left(\frac{1}{16} M\left(\frac{r}{2^2}, f_{n-2}\right), g\right) + O(1).$$

Taking repeated logarithms p times, we get

$$\begin{aligned} \log^{[2p+2-q]} M(r, f_n) &\geq \log^{[p+1]} M\left(\frac{1}{16} M\left(\frac{r}{2^2}, f_{n-2}\right), g\right) + O(1) \\ &\geq \left(\lambda_g(p, q) - \varepsilon \right) \log^{[q]} \left[\frac{1}{16} M\left(\frac{r}{2^2}, f_{n-2}\right) \right] + O(1) \end{aligned}$$

$$\log^{[2p+3-2q]} M(r, f_n) \geq \log M\left(\frac{r}{2^2}, f_{n-2}\right) + O(1).$$

Again taking repeated logarithms p times, we get

$$\begin{aligned} \log^{[2p+2-2q]} M(r, f_n) &\geq \left(\lambda_f(p, q) - \varepsilon \right) \log^{[q]} \left[\frac{1}{16} M\left(\frac{r}{2^3}, g_{n-3}\right) \right] + O(1) \\ &\geq \left(\lambda_f(p, q) - \varepsilon \right) \log^{[q]} M\left(\frac{r}{2^3}, g_{n-3}\right) + O(1). \end{aligned}$$

Finally, after taking repeated logarithms $(n-4)(p+1)$ times more, we have for all sufficiently large values of r ,

$$\begin{aligned} \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) &\geq \left(\lambda_f(p, q) - \varepsilon \right) \log^{[q]} \left[\frac{1}{16} M\left(\frac{r}{2^{n-1}}, g\right) \right] + O(1) \\ \text{i.e., } \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) &\geq \left(\lambda_f(p, q) - \varepsilon \right) \log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) + O(1). \end{aligned}$$

Similarly if n is odd then for all sufficiently large values of r ,

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \geq \left(\lambda_g(p, q) - \varepsilon \right) \log^{[q]} M\left(\frac{r}{2^{n-1}}, f\right) + O(1).$$

This proves the lemma.

Lemma 2.6. Let $f(z)$ and $g(z)$ be two non-constant entire functions, such that

$0 < \rho_f(p, q) < \infty$ and $0 < \rho_g(p, q) < \infty$. Then for all sufficiently large r and $\varepsilon > 0$,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \leq \begin{cases} \left(\rho_f(p, q) + \varepsilon \right) \log^{[q]} M(r, g) + O(1) & \text{when } n \text{ is even,} \\ \left(\rho_g(p, q) + \varepsilon \right) \log^{[q]} M(r, f) + O(1) & \text{when } n \text{ is odd} \end{cases} \quad \text{where}$$

$$p \geq q \geq 1.$$

The lemma follows from Lemma 2.1 and Lemma 2.4.

Lemma 2.7. Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_f(p, q) < \infty$ and $0 < \lambda_g(p, q) < \infty$. Then for any $\varepsilon (0 < \varepsilon < \min\{\lambda_f(p, q), \lambda_g(p, q)\})$ and $p \geq q \geq 1$,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \geq \begin{cases} (\lambda_f(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right) + O(1) & \text{when } n \text{ is even,} \\ (\lambda_g(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^{n-1}}, f\right) + O(1) & \text{when } n \text{ is odd} \end{cases}$$

for all sufficiently large values of r .

Proof. To prove this lemma we first suppose that n is even. Then from Lemma 2.1 and Lemma 2.3 we get for any $\varepsilon (0 < \varepsilon < \min\{\lambda_f(p, q), \lambda_g(p, q)\})$ and for all sufficiently large values of r ,

$$\begin{aligned} T(r, f_n) &= T\left(r, f(g_{n-1})\right) \\ &\geq \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4}, g_{n-1}\right) + O(1), f\right). \end{aligned}$$

$$\begin{aligned} \therefore \log^{[p]} T(r, f_n) &\geq \log^{[p+1]} M\left(\frac{1}{8} M\left(\frac{r}{4}, g_{n-1}\right) + O(1), f\right) + O(1) \\ &\geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} \left[\frac{1}{8} M\left(\frac{r}{4}, g_{n-1}\right) + O(1) \right] + O(1) \\ &\geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} \left[\frac{1}{9} M\left(\frac{r}{4}, g_{n-1}\right) \right] + O(1) \\ &\geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4}, g_{n-1}\right) + O(1) \\ &\geq (\lambda_f(p, q) - \varepsilon) \log^{[q-1]} T\left(\frac{r}{4}, g_{n-1}\right) + O(1) \\ &\geq (\lambda_f(p, q) - \varepsilon) \log^{[q-1]} \left[\frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) \right] + O(1), \end{aligned}$$

$$\text{i.e., } \log^{[p+1]} T(r, f_n) \geq \log^{[q+1]} M\left(\frac{1}{8} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + O(1)$$

$$\text{i.e., } \log^{[p+1-q]} T(r, f_n) \geq \log M\left(\frac{1}{8} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + O(1)$$

$$\begin{aligned} \text{i.e., } \log^{[2p+1-q]} T(r, f_n) &\geq \log^{[p+1]} M\left(\frac{1}{8} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1), g\right) + O(1) \\ &\geq (\lambda_g(p, q) - \varepsilon) \log^{[q]} \left[\frac{1}{8} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1) \right] + O(1) \\ &\geq (\lambda_g(p, q) - \varepsilon) \log^{[q]} \left[\frac{1}{9} M\left(\frac{r}{4^2}, f_{n-2}\right) \right] + O(1). \end{aligned}$$

$$\text{i.e., } \log^{[2p+1-q]} T(r, f_n) \geq (\lambda_g(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^2}, f_{n-2}\right) + O(1)$$

$$\begin{array}{ccc} \dots & \dots & \dots \\ \dots & \dots & \dots \end{array}$$

$$\therefore \log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right) + O(1) \text{ when } n \text{ is even. Si}$$

milarly,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \geq (\lambda_g(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^{n-1}}, f\right) + O(1) \text{ when } n \text{ is odd.}$$

This proves the lemma.

3. Theorems

Theorem 3.1. Let f and g be two non-constant entire functions of non-zero finite (p, q) -th order and lower (p, q) -th order, also $0 < \sigma_f^q, \sigma_g^q < \infty$. Then

$$\begin{aligned} (i) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} &\leq \frac{4^{\rho_g^q} \rho_f(p, q)}{\lambda_f(p, q)}, \\ (ii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} &\geq \frac{\lambda_f(p, q)}{(2^{n-1})^{\rho_g^q} \rho_f(p, q)} \end{aligned}$$

when n is even and

$$\begin{aligned} (iii) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} &\leq \frac{4^{\rho_f^q} \rho_g(p, q)}{\lambda_g(p, q)}, \\ (iv) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} &\geq \frac{\lambda_g(p, q)}{(2^{n-1})^{\rho_f^q} \rho_g(p, q)} \end{aligned}$$

when n is odd.

Proof. First we suppose that n is even, then from Lemma 2.4 and the Definition 1.5 we have for all large r and $\varepsilon > 0$,

$$\begin{aligned} \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) &\leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1) \\ &\leq (\rho_f(p, q) + \varepsilon) (\sigma_g^q + \varepsilon) r^{\rho_g^q} + O(1). \end{aligned} \quad (3.1)$$

From Lemma 2.3 we get

$$T(r, f(g)) \geq \frac{1}{3} \log M\left(\frac{1}{16} M\left(\frac{r}{4}, g\right), f\right).$$

Since $\lambda_f(p, q)$ is the lower (p, q) -th order of f so for given $\varepsilon (0 < \varepsilon < \lambda_f(p, q))$ and for all large values of r ,

$$\begin{aligned} \log^{[p]} T(r, f(g)) &\geq \log^{[p+1]} M\left(\frac{1}{16} M\left(\frac{r}{4}, g\right), f\right) + O(1) \\ \therefore \log^{[p]} T(r, f(g)) &\geq \left(\lambda_f(p, q) - \varepsilon\right) \log^{[q]} M\left(\frac{r}{4}, g\right) + O(1). \end{aligned} \quad (3.2)$$

Again for a sequence of values of r tending to infinity,

$$\log^{[q]} M\left(\frac{r}{4}, g\right) > \left(\sigma_g^q - \varepsilon\right) \left(\frac{r}{4}\right)^{\rho_g^q}. \quad (3.3)$$

Therefore from (3.2) and (3.3) we get for a sequence of values of r tending to infinity,

$$\log^{[p]} T(r, f(g)) \geq \left(\lambda_f(p, q) - \varepsilon\right) \left(\sigma_g^q - \varepsilon\right) \left(\frac{r}{4}\right)^{\rho_g^q} + O(1) \quad (3.4)$$

where $0 < \varepsilon < \min\{\lambda_f(p, q), \sigma_g^q\}$.

Now from (3.1) and (3.4) we have for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} &\leq \frac{(\rho_f(p, q) + \varepsilon)(\sigma_g^q + \varepsilon) r^{\rho_g^q} + O(1)}{(\lambda_f(p, q) - \varepsilon)(\sigma_g^q - \varepsilon) \left(\frac{r}{4}\right)^{\rho_g^q} + O(1)} \\ &= \frac{(\rho_f(p, q) + \varepsilon)(\sigma_g^q + \varepsilon) 4^{\rho_g^q} + o(1)}{(\lambda_f(p, q) - \varepsilon)(\sigma_g^q - \varepsilon) + o(1)}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \leq \frac{4^{\rho_g^q} \rho_f(p, q)}{\lambda_f(p, q)}.$$

Also when n is even then from Lemma 2.5 we get for all sufficiently large values of r

$$\log^{[(n-1)(p-1)-(n-2)q]} M(r, f_n) \geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) + O(1).$$

Now for a sequence of values of r tending to infinity, we have

$$\log^{[q]} M\left(\frac{r}{2^{n-1}}, g\right) > \left(\sigma_g^q - \varepsilon\right) \left(\frac{r}{2^{n-1}}\right)^{\rho_g^q}.$$

Therefore for a sequence of values of r tending to infinity, we get

$$\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) \geq (\lambda_f(p, q) - \varepsilon) \left(\sigma_g^q - \varepsilon\right) \left(\frac{r}{2^{n-1}}\right)^{\rho_g^q} + O(1) \quad (3.5) \text{ where}$$

$$0 < \varepsilon < \min\{\lambda_f(p, q), \sigma_g^q\}.$$

Again by Lemma 2.1 we have for all large values of r , and $\varepsilon > 0$,

$$\begin{aligned} T(r, f(g)) &\leq \log M(r, f(g)) \\ &\leq \log M(M(r, g), f). \end{aligned}$$

$$\begin{aligned} \therefore \log^{[p]} T(r, f(g)) &\leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) \\ &\leq (\rho_f(p, q) + \varepsilon) (\sigma_g^q + \varepsilon) r^{\rho_g^q}. \end{aligned} \quad (3.6)$$

Therefore from (3.5) and (3.6) we have for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} &\geq \frac{(\lambda_f(p, q) - \varepsilon) (\sigma_g^q - \varepsilon) \left(\frac{r}{2^{n-1}}\right)^{\rho_g^q} + O(1)}{(\rho_f(p, q) + \varepsilon) (\sigma_g^q + \varepsilon) r^{\rho_g^q}} \quad \text{Since} \\ &= \frac{(\lambda_f(p, q) - \varepsilon) (\sigma_g^q - \varepsilon) + o(1)}{(2^{n-1})^{\rho_g^q} (\rho_f(p, q) + \varepsilon) (\sigma_g^q + \varepsilon)}. \end{aligned}$$

$\varepsilon > 0$ is arbitrary,

$$\therefore \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} \geq \frac{\lambda_f(p, q)}{(2^{n-1})^{\rho_g^q} \rho_f(p, q)}.$$

Similarly for odd n we get the second part of this theorem.

This proves the theorem.

Remark 3.2 If f is of regular growth i.e. $\rho_f(p, q) = \lambda_f(p, q)$ and n is even then

$$\begin{aligned} (i) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} &\leq 4^{\rho_g^q}, \\ (ii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} &\geq \frac{1}{(2^{n-1})^{\rho_g^q}}. \end{aligned}$$

Also if g is of regular growth i.e. $\rho_g(p, q) = \lambda_g(p, q)$ and n is odd then

$$\begin{aligned} (iii) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} &\leq 4^{\rho_f^q}, \\ (iv) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} &\geq \frac{1}{(2^{n-1})^{\rho_f^q}}. \end{aligned}$$

Remark 3.3. The conditions non zero lower (p, q) -th order and finite (p, q) -th order are necessary for Theorem 3.1, which are shown by the following examples.

Example 3.4. Let $f(z) = \exp^{[p-q+1]} z$, $g(z) = \exp^{[p-q]} z$ and $2q \geq p+1$. Then

$$\rho_f(p, q) = \lambda_f(p, q) = 1 \text{ and } \rho_g(p, q) = \lambda_g(p, q) = 0.$$

Here $f(g) = \exp^{[2p-2q+1]} z$ and

$$\begin{aligned}
3T(2r, f(g)) &\geq \log M(r, f(g)) = \exp^{[2p-2q]}_r \\
\text{i.e., } T(r, f(g)) &\geq \frac{1}{3} \exp^{[2p-2q]}_r \frac{r}{2}. \\
\therefore \log^{[p]} T(r, f(g)) &\geq \exp^{[p-2q]}_r \frac{r}{2} + O(1).
\end{aligned}$$

Now

$$f_n = \begin{cases} \exp^{[n p - n q + \frac{n}{2}]}_z & \text{when } n \text{ is even} \\ \exp^{[n p - n q + \frac{n+1}{2}]}_z & \text{when } n \text{ is odd.} \end{cases}$$

So when n is even,

$$\begin{aligned}
M(r, f_n) &= \exp^{[n p - n q + \frac{n}{2}]}_r \\
\therefore \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) &= \log^{[(n-1)(p+1)-(n-2)q]} \exp^{[n p - n q + \frac{n}{2}]}_r \\
&= \exp^{[p-2q-\frac{n}{2}+1]}_r.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, f(g))} &\leq \frac{\exp^{[p-2q-\frac{n}{2}+1]}_r}{\exp^{[p-2q]}_r} + o(1) \\
&= \frac{1}{\exp^{[\frac{n-1}{2}]}_r} + o(1) \rightarrow 0 \text{ not gater then } 1 \text{ as } r \rightarrow \infty.
\end{aligned}$$

Similarly for odd n ,

$$\begin{aligned}
\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} &\leq \frac{\exp^{[p-2q-\frac{n-1}{2}+1]}_r}{\exp^{[p-2q]}_r} + o(1) \\
&\rightarrow 0 \text{ not gater then } 1 \text{ as } r \rightarrow \infty.
\end{aligned}$$

Example 3.5. Let $f(z) = \exp^{[p-q+1]}_z$, $g(z) = \exp^{[p-q+2]}_z$ and $2q \geq p+1$. Then $\rho_f(p, q) = \lambda_f(p, q) = 1$ and $\rho_g(p, q) = \lambda_g(p, q) = \infty$.

Here $g(f) = \exp^{[2p-2q+3]}_z$ and

$$\begin{aligned}
T(r, g(f)) &\leq \log M(r, g(f)) = \exp^{[2p-2q+2]}_r \\
\therefore \log^{[p]} T(r, g(f)) &\leq \exp^{[p-2q+2]}_r.
\end{aligned}$$

Now

$$f_n = \begin{cases} \exp\left[n p - nq + \frac{3n}{2}\right] z & \text{when } n \text{ is even} \\ \exp\left[n p - nq + \frac{3n-1}{2}\right] z & \text{when } n \text{ is odd.} \end{cases}$$

So when n is even,

$$M(r, f_n) = \exp\left[n p - nq + \frac{3n}{2}\right]_r$$

$$\begin{aligned} \text{i.e., } \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) &= \log^{[(n-1)(p+1)-(n-2)q]} \exp\left[n p - nq + \frac{3n}{2}\right]_r \\ &= \exp\left[p-2q+\frac{n+1}{2}\right]_r. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} &\geq \frac{\exp\left[p-2q+\frac{n+1}{2}\right]_r}{\exp\left[p-2q+2\right]_r} \\ &= \exp\left[\frac{n-1}{2}\right]_r \rightarrow \infty \text{ not less than } 1 \text{ as } r \rightarrow \infty. \end{aligned}$$

When n is odd,

$$M(r, f_n) = \exp\left[n p - nq + \frac{3n-1}{2}\right]_r$$

$$\begin{aligned} \text{i.e., } \log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n) &= \log^{[(n-1)(p+1)-(n-2)q]} \exp\left[n p - nq + \frac{3n-1}{2}\right]_r \\ &= \exp\left[p-2q+\frac{n+1}{2}\right]_r. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[p]} T(r, g(f))} &\geq \frac{\exp\left[p-2q+\frac{n+1}{2}\right]_r}{\exp\left[p-2q+2\right]_r} \\ &= \exp\left[\frac{n+1}{2}-2\right]_r \rightarrow \infty \text{ not less than } 1 \text{ as } r \rightarrow \infty. \end{aligned}$$

Theorem 3.6. Let f and g be two non-constant entire functions of non-zero finite (p, q) -th order and lower (p, q) -th order, also $0 < \sigma_f^q, \sigma_g^q < \infty$. Then

$$(i) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \frac{(4^{n-1})^{\rho_g^q} \rho_f(p, q)}{\lambda_f(p, q)},$$

$$(ii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{\lambda_f(p, q)}{2^{\rho_g^q} \rho_f(p, q)}$$

when n is even and

$$(iii) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, g(f))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \frac{(4^{n-1})^{\rho_f^q} \rho_g(p, q)}{\lambda_g(p, q)},$$

$$(iv) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{\lambda_g(p, q)}{2^{\rho_f^q} \rho_g(p, q)}$$

when n is odd.

Proof. when n is even, then from Lemma 2.7, we get for all large values of r and any $\varepsilon (0 < \varepsilon < \lambda_f(p, q))$,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \geq (\lambda_f(p, q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right) + O(1). \quad (3.7)$$

Again for a sequence of values of r tending to infinity,

$$\log^{[q]} M\left(\frac{r}{4^{n-1}}, g\right) > (\sigma_g^q - \varepsilon) \left(\frac{r}{4^{n-1}}\right)^{\rho_g^q}. \quad (3.8)$$

Therefore from (3.7) and (3.8) we have for a sequence of values of r tending to infinity,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \geq (\lambda_f(p, q) - \varepsilon) (\sigma_g^q - \varepsilon) \left(\frac{r}{4^{n-1}}\right)^{\rho_g^q} + O(1) \quad (3.9) \text{ where}$$

$$0 < \varepsilon < \min\{\lambda_f(p, q), \sigma_g^q\}.$$

Now from second part of Lemma 2.2 we get for large values of r ,

$$\begin{aligned} \log^{[p+1]} M(r, f(g)) &\leq \log^{[p+1]} M(M(r, g), f) \\ &\leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) \\ &\leq (\rho_f(p, q) + \varepsilon) (\sigma_g^q + \varepsilon) r^{\rho_g^q}. \end{aligned} \quad (3.10)$$

Now from (3.9) and (3.10) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \frac{(\rho_f(p, q) + \varepsilon)(\sigma_g^q + \varepsilon)r^{\rho_g^q}}{(\lambda_f(p, q) - \varepsilon)(\sigma_g^q - \varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\rho_g^q} + O(1)}$$

$$= \frac{(\rho_f(p, q) + \varepsilon)(\sigma_g^q + \varepsilon)(4^{n-1})^{\rho_g^q}}{(\lambda_f(p, q) - \varepsilon)(\sigma_g^q - \varepsilon) + o(1)}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \frac{(4^{n-1})^{\rho_g^q} \rho_f(p, q)}{\lambda_f(p, q)}.$$

Again for all sufficiently large values of r , we get from first part of Lemma 2.2,

$$\log^{[p+1]} M(r, f(g)) \geq \log^{[p+1]} M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right)$$

$$\geq \left(\lambda_f(p, q) - \varepsilon\right) \log^{[q]} M\left(\frac{r}{2}, g\right) + o(1). \quad (3.11)$$

Also for a sequence of values of r tending to infinity,

$$\log^{[q]} M\left(\frac{r}{2}, g\right) > (\sigma_g^q - \varepsilon)\left(\frac{r}{2}\right)^{\rho_g^q}. \quad (3.12)$$

Therefore from (3.11) and (3.12) for a sequence of values of r tending to infinity,

$$\log^{[p+1]} M(r, f(g)) \geq \left(\lambda_f(p, q) - \varepsilon\right)(\sigma_g^q - \varepsilon)\left(\frac{r}{2}\right)^{\rho_g^q} + O(1) \quad (3.13)$$

where $0 < \varepsilon < \min\{\lambda_f(p, q), \sigma_g^q\}$.

Also when n is even then from Lemma 2.6 we get for r tending to infinity,

$$\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n) \leq (\rho_f(p, q) + \varepsilon) \log^{[q]} M(r, g) + O(1)$$

$$\leq (\rho_f(p, q) + \varepsilon)(\sigma_g^q + \varepsilon)r^{\rho_g^q} + O(1). \quad (3.14)$$

Now from (3.13) and (3.14) for a sequence of values of r tending to infinity,

$$\frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{(\lambda_f(p, q) - \varepsilon)(\sigma_g^q - \varepsilon)\left(\frac{r}{2}\right)^{\rho_g^q} + O(1)}{(\rho_f(p, q) + \varepsilon)(\sigma_g^q + \varepsilon)r^{\rho_g^q} + O(1)}$$

$$= \frac{(\lambda_f(p, q) - \varepsilon)(\sigma_g^q - \varepsilon) + o(1)}{2^{\rho_g^q} (\rho_f(p, q) + \varepsilon)(\sigma_g^q + \varepsilon) + o(1)}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\therefore \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{\lambda_f(p, q)}{2^{\rho_g^q} \rho_f(p, q)}.$$

Similarly for odd n we get the second part of this theorem.
This proves the theorem.

Remark 3.7. If f is of regular growth i.e. $\rho_f(p, q) = \lambda_f(p, q)$ and n is even then

$$(i) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq (4^{n-1})^{\rho_g^q},$$

$$(ii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, f(g))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{1}{2^{\rho_g^q}}.$$

Also if g is of regular growth i.e. $\rho_g(p, q) = \lambda_g(p, q)$ and n is odd then

$$(iii) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, g(f))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq (4^{n-1})^{\rho_f^q},$$

$$(iv) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[p+1]} M(r, g(f))}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{1}{2^{\rho_f^q}}.$$

The next theorem is the generalization of the above theorems.

Theorem 3.8. Let f and g be two non-constant entire functions of non-zero finite (p, q) -th order and lower (p, q) -th order, also $0 < \sigma_f^q, \sigma_g^q < \infty$. Then

$$(i) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \frac{(4^{n-1})^{\rho_g^q} \rho_f(p, q)}{\lambda_f(p, q)},$$

$$(ii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{\lambda_f(p, q)}{(2^{n-1})^{\rho_g^q} \rho_f(p, q)}$$

when n is even and

$$\begin{aligned}
(iii) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} &\leq \frac{(4^{n-1})^{\rho_f^q} \rho_g(p, q)}{\lambda_g(p, q)}, \\
(iv) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} &\geq \frac{\lambda_g(p, q)}{(2^{n-1})^{\rho_f^q} \rho_g(p, q)}
\end{aligned}$$

where

n is odd.

Proof. When n is even then from (3.1) and (3.9) we get for a sequence of values of r tending to infinity and for $0 < \varepsilon < \min \left\{ \lambda_f(p, q), \sigma_g^q \right\}$

$$\begin{aligned}
\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} &\leq \frac{\left(\rho_f(p, q) + \varepsilon \right) \left(\sigma_g^q + \varepsilon \right) r^{\rho_g^q} + O(1)}{\left(\lambda_f(p, q) - \varepsilon \right) \left(\sigma_g^q - \varepsilon \right) \left(\frac{r}{4^{n-1}} \right)^{\rho_g^q} + O(1)} \quad \text{Sinc} \\
&= \frac{\left(\rho_f(p, q) + \varepsilon \right) \left(\sigma_g^q + \varepsilon \right) (4^{n-1})^{\rho_g^q} + o(1)}{\left(\lambda_f(p, q) - \varepsilon \right) \left(\sigma_g^q - \varepsilon \right) + o(1)}.
\end{aligned}$$

is $\varepsilon > 0$ arbitrary,

$$\therefore \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq \frac{(4^{n-1})^{\rho_g^q} \rho_f(p, q)}{\lambda_f(p, q)}.$$

Also from (3.5) and (3.14) we have for a sequence of values of r tending to infinity and for

$$0 < \varepsilon < \left\{ \lambda_f(p, q), \sigma_g^q \right\},$$

$$\begin{aligned}
\frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} &\geq \frac{\left(\lambda_f(p, q) - \varepsilon \right) \left(\sigma_g^q - \varepsilon \right) \left(\frac{r}{2^{n-1}} \right)^{\rho_g^q} + O(1)}{\left(\rho_f(p, q) + \varepsilon \right) \left(\sigma_g^q + \varepsilon \right) r^{\rho_g^q} + O(1)} \\
&= \frac{\left(\lambda_f(p, q) - \varepsilon \right) \left(\sigma_g^q - \varepsilon \right) + o(1)}{\left(\rho_f(p, q) + \varepsilon \right) \left(\sigma_g^q + \varepsilon \right) (2^{n-1})^{\rho_g^q} + o(1)}.
\end{aligned}$$

Since is $\varepsilon > 0$ arbitrary,

$$\therefore \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{\lambda_f(p, q)}{(2^{n-1})^{\rho_g^q} \rho_f(p, q)}.$$

Similarly for odd n we get the second part of the theorem.
This proves the theorem.

Remark 3.9. If f is regular growth i.e. $\rho_f(p, q) = \lambda_f(p, q)$ and n is even then

$$(i) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq (4^{n-1})^{\rho_g^q},$$

$$(ii) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{1}{(2^{n-1})^{\rho_g^q}}.$$

Also if g is regular growth i.e. $\rho_g(p, q) = \lambda_g(p, q)$ and n is odd then

$$(iii) \quad \liminf_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \leq (4^{n-1})^{\rho_f^q},$$

$$(iv) \quad \limsup_{r \rightarrow \infty} \frac{\log^{[(n-1)(p+1)-(n-2)q]} M(r, f_n)}{\log^{[(n-1)p+(n-2)(1-q)]} T(r, f_n)} \geq \frac{1}{(2^{n-1})^{\rho_f^q}}.$$

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