

Biorthogonal Wavelet Packets Associated with Nonuniform Multiresolution

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Abstract. In this paper, we introduce the notion of biorthogonal wavelet packets associated with nonuniform multiresolution analysis and study their characteristics by means of Fourier transform. Three biorthogonal formulas regarding these wavelet packets are established. Moreover, it is shown how to obtain several new Riesz bases of the space $L^2(\mathbb{R})$ by constructing a series of subspaces of these nonuniform wavelet packets.

Keywords: Wavelet; nonuniform multiresolution analysis; biorthogonal wavelet packet; Riesz basis; Fourier transform.

1. Introduction

In his pioneering paper, Mallat [11] first formulated a new and remarkable idea of multiresolution analysis (MRA) which deals with a general formalism for the construction of an orthonormal basis of wavelet bases. Mathematically, an MRA consists of a sequence of embedded closed subspaces, $\{V_j: j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that $f \in V_j$ if and only if $f(2x) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of function φ , $\{\varphi(x - k): k \in \mathbb{Z}\}$ represents a complete orthonormal system for V_0 . The function φ is called the *scaling function* or the *father wavelet*. Recently, the idea of MRA and wavelets have been generalized in many different settings, for example, one can replace the dilation factor 2 by an integer $M \geq 2$ and in higher dimensions, it can be replaced by a dilation matrix A , in which case the number of wavelets required is $|\det A| - 1$. But in all these cases, the translation set is always a group. In the two papers [6, 7], Gabardo and Nashed considered a generalization of Mallat's [11] celebrated theory of MRA based on spectral pairs, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. More precisely, this set is of the form $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N - 1$, r is an odd integer relatively prime to N . They call this a *nonuniform multiresolution analysis* (NUMRA). Moreover, they have provided the necessary and sufficient conditions for the existence of associated wavelets in $L^2(\mathbb{R})$. Later on, Gabardo and Yu [8, 9] continued their research in this non-standard setting and gave the characterization of all nonuniform wavelets associated with nonuniform multiresolution analysis.

It is well-known that the classical orthonormal wavelet bases have poor frequency localization. To overcome this disadvantage, Coifman *et al.* [5] constructed univariate orthogonal wavelet packets as a generalization of Walsh systems. Wavelet packets give rise to a large class of orthonormal bases of $L^2(\mathbb{R})$, each one corresponding to a different splitting of $L^2(\mathbb{R})$ into direct sum of its closed subspaces. Wavelet packets, due to their nice characteristics have been widely applied to signal processing, coding theory, image compression, fractal theory and solving integral equations and so on. Chui and Li [3] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. The introduction of biorthogonal wavelet packets attributes to Cohen and Daubechies [4]. Shen [14] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the orthogonal version of vector-valued wavelet packets [2], higher dimensional wavelet packets with arbitrary dilation matrix [10], the orthogonal p -wavelet packets and the p -wavelet frame packets on the positive half-line $\mathbb{R}^+$ [12, 13].

In his recent paper, Behera [1] has constructed nonuniform wavelet packets associated with nonuniform multiresolution analysis. He proved lemmas on the so-called splitting trick and several theorems

concerning the construction of nonuniform wavelet packets on $\mathbb{R}$. It is well known that the orthogonal wavelet packets have many desired properties such as compact support, good frequency localization and vanishing moments. However, there is no continuous symmetry which is a much desired property in image and signal processing. To achieve symmetry, several generalizations of scalar orthogonal wavelet packets have been investigated in literature. The biorthogonal wavelet packets achieve symmetry where the orthogonality is replaced by the biorthogonality. Therefore, the main goal of this paper is to introduce the notion of biorthogonal wavelet packets associated with nonuniform multiresolution analysis and investigate their properties by means of the Fourier transform. Further, we also construct several new Riesz bases of space $L^2(\mathbb{R})$ by constructing a series of subspaces of nonuniform wavelet packets.

2. Nonuniform multiresolution analysis and the wavelet packets

In this section, we state some basic preliminaries and definitions including nonuniform multiresolution analysis, the associated nonuniform wavelets and wavelet packets. More details are referring to [6–9].

Definition 2.1. We say that a pair of functions $f(x), \tilde{f}(x) \in L^2(\mathbb{R})$ are *biorthogonal*, if their translates satisfy

$$\langle f(\cdot), \tilde{f}(\cdot - \lambda) \rangle = \delta_{0,\lambda}, \quad \lambda \in \Lambda, \quad (2.1)$$

where $\delta_{0,\lambda}$ is *Kronecker symbol*, i.e., $\delta_{0,\lambda} = 1$ when $\lambda = 0$ and $\delta_{0,\lambda} = 0$, otherwise.

Definition 2.2. Let $\mathbb{H}$ be a Hilbert space. A sequence $\{f_k\}_{k=1}^{\infty}$ of $\mathbb{H}$ is said to be a Riesz basis for $\mathbb{H}$ if there exist constants A and $B, 0 < A \leq B < \infty$ such that any $f \in \mathbb{H}$ can be represented as a series $f = \sum_{k=1}^{\infty} c_k f_k$ converging in $\mathbb{H}$ with

$$A \|f\|_2^2 \leq \sum_{k=1}^{\infty} |c_k|^2 \leq B \|f\|_2^2. \quad (2.2)$$

We first recall the definition of nonuniform multiresolution analysis and some of its properties.

Definition 2.3. A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces $L^2(\mathbb{R})$ is called a *nonuniform multiresolution analysis* of $L^2(\mathbb{R})$ if the following hold:

- (i) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$,
- (ii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,
- (iii) $f(x) \in V_j$ if and only if $f(2N \cdot) \in V_{j+1}$,
- (iv) there is a function φ in V_0 , called the *scaling function*, such that the system of functions $\{\varphi(\cdot - \lambda) : \lambda \in \Lambda\}$ forms a Riesz basis for subspace V_0 .

It is worth noticing that, when $N = 1$, one recovers from the definition above the standard definition of a one-dimensional multiresolution analysis with dilation factor equal to 2. When $N > 1$, the dilation factor of $2N$ ensures that $2N\Lambda \subset 2\mathbb{Z} \subset \Lambda$.

Since $\varphi(x) \in V_0 \subset V_1$, there exists sequence $\{a_\lambda\}_{\lambda \in \Lambda}$ with $\sum_{\lambda \in \Lambda} |a_\lambda|^2 < \infty$ such that

$$\varphi\left(\frac{x}{2N}\right) = \sum_{\lambda \in \Lambda} a_\lambda \varphi(x - \lambda). \quad (2.3)$$