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Matrix technology for a class of fourth-order difference schemes in solution of hyperbolic equations

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Abstract. In this article, We apply Krylov subspace methods in combination of the ADI, BLAGE,... method as a preconditioner for a class of linear systems arising from fourth-order finite difference schemes in solution of hyperbolic equations $\alpha u_{tt} - \beta(x,t)u_{xx} = F(x,t,u,u_x,u_t)$ subject to appropriate initial and Dirichlet boundary conditions, where α is constant. We show The BLAGE preconditioner is extremely effective in achieving optimal convergence rates for the class of fourth-order difference schemes considered in this paper. Numerical results performed on model problem to confirm the efficiency of our approach.

Keywords: Fourth-order approximation; Hyperbolic equations; Krylov subspace methods; Preconditioner.

1. Introduction

In Solution of PDE's by means of numerical methods one often has to deal with large systems of linear equations, especially if the PDE's is time-independent or if the time-integrator is implicit. For real life problems, these large systems can often only be solved by means of some iterative method. Even if the system are preconditioned, the basic iterative method often converges slowly or even diverges. The numerical solution of one space second order hyperbolic equations with nonlinear first derivative terms in Cartesian, cylindrical and spherical coordinates are of great importance in many fields of engineering and sciences.

Many computational models give rise to large sparse linear systems. For such systems iterative methods are usually preferred to direct methods which are expensive both in memory and computing requirements. When the iterative method is based on Krylov subspaces, there is a need to use preconditioning techniques in order to achieve convergence in a reasonable number of iteration steps. Since the preconditioner plays a critical role in preconditioned Krylov subspace methods, many preconditioners have been proposed and studied [22, 5, 11]. These, preconditioners based on incomplete factorization such as ILU preconditioner that have been proposed and studied by many of researchers [17, 10, 11]. The ADI method is a preconditioner for non-symmetric systems that can be very effective but this method is not effective for more general block tri-diagonal systems arising from the fourth-order approximations. Bhuruth and Evans [3] proposed BLAGE method as a preconditioner for a class of non-symmetric linear systems arising from the fourth-order finite difference schemes. In this article, we compare different preconditioned methods for solving linear systems arising from the fourth-order approximation of hyperbolic equation

$$\alpha u_{tt} - \beta(x,t)u_{xx} = F(x,t,u,u_x,u_t)$$

defined in the region $W \times [0 < t < T]$, where $W = \{x \mid 0 < x < 1\}$ and α is constant. The initial and boundary conditions consists of

$$u(x,0) = g_1(x), \qquad u_t(x,0) = g_2(x), \qquad 0 \le x \le 1,$$

$$u(0,t) = h_0(t), \qquad u(1,t) = h_1(t), \qquad t \ge 0,$$

Where $u = u(x,t)$. The resulting block tri-diagonal linear system of equations is solved by using Krylov subspace methods. The outline of this paper is as follows: In Section 2, we describe Krylov subspace methods. In Section 3, we briefly introduce some available preconditioners. In Section 4 we present a class of fourth-order finite difference operators and in Section 5, we present an example to illustrate the accuracy of our method. In Section 6, we report a brief conclusion.

2. Krylov subspace methods

Let x_0 be an arbitrary initial guess for linear systems given by $Ax = b$ and let $r_0 = b - Ax_0$ be the corresponding residual vector. A Krylov subspace of order m that is shown with $K_m(A, r)$ is defined as follows:

$$K_m(A, r_0) = \text{span}\{r_0, A r_0, ..., A^{m-1} r_0\}.$$

For un-symmetric matrix A, different Krylov methods can be used such as GMRES, GMRES(m), QMR, CGS, BiCG, BiCGSTAB [18, 24]. Now, we briefly describe some Krylov subspace methods:

2.1. Generalized Minimal Residual(GMRES) method

In 1986, Saad and Schultz [19] introduced GMRES method for solving non-symmetric systems. This method has the property of minimizing the norm of the residual vector over the Krylov subspace method at every step. The major drawback for GMRES method is that the amounts of work and storage required per iteration linearly rises with the iteration number. The usual way for overcome this problem is to restart after m iteration.

Proposition 2.1. Assume that A is a diagonalizable matrix and let $A = XDX^{-1}$ where $D = \text{diag}\{\lambda_1, ..., \lambda_n\}$ is the diagonal matrix of eigenvalues. Define,

$$\varepsilon^m = \min\_\{p \varepsilon P_m, p(0) = 1\} \max\_\{i = 1, ..., n\} | p(\lambda_i) |.$$

Then, the residual norm achieved by the m-th step of GMRES satisfies the inequality

$$\| r_m \|_2 \leq K(X) \grave{o}^m \| r_0 \|_2,$$

where , $K(X) = \| X \|_2 \| X^{-1} \|_2$.

If A is positive real with symmetric part M , the following error bound can be derived from the proposition,

$$\| r_m \| \leq [1 - \alpha / \beta]^{m/2} \| r_0 \|,$$

with $\alpha = (\lambda_{\min}(M))^2$, $\beta = \lambda_{\max}(A^T A)$. This proves the convergence of the GMRES(m) for all m when A is positive real [18].

2.2. Bi-Conjugate Gradient (BiCG) method